Theorem 6

$$\sum_{\{x,y\}\in E} \frac{R_{xy}}{r_{xy}} = n-1.$$

Proof. The proof is similar to the proof of Theorem 3, and begins with the reciprocity relation,

$$U_z^{xy}/c(z) = U_x^{zy}/c(x)$$

Multiplying both sides by c_{xy}, and taking sums on both sides,

$$\begin{aligned}
\sum_{x\in N(y)} U_z^{xy} c_{xy}/c(z) = \sum_{x\in N(y)} U_x^{zy} c_{xy}/c(x) &= 1 \text{ if } z \neq y \\
\Rightarrow \sum_{y\in G}\sum_{x\in N(y)} U_z^{xy} c_{xy}/c(z) &= n-1 \\
\Rightarrow \sum_{\{x,y\}\in G} \Big(U_z^{xy} c_{xy}/c(z) + U_z^{yx} c_{yx}/c(z) \Big) &= n-1
\end{aligned}$$

This clearly implies $\sum_{\{x,y\}\in E} \frac{R_{xy}}{r_{xy}} = n-1$.

Acknowledgements. We thank Peter G. Doyle for informing us of Foster's work. Thanks also to Prabhakar Raghavan and Larry Ruzzo for their comments and critical remarks.

References

[1] Chandra A.K., Raghavan P., Ruzzo W.L., Smolensky R., and Tiwari P., The Electrical Resistance of a Graph Captures its Commute and Cover Times, *Proceedings of the 21st Annual ACM Symposium on Theory of Computing*, May (1989).

[2] Doyle P.G. and Snell J.L., *Random Walks and Electrical Networks*, The Mathematical Association of America (1984).

[3] Foster R.M., The Average Impedance of an Electrical Network, *Contributions to Applied Mechanics* (Reissner Anniversary Volume), Ann Arbor (1949), pp. 333-340.

[4] Göbel F. and Jagers A.A., Random Walks on Graphs, *Stochastic Processes and their Applications*, **2** (1974), pp. 311-336.

[5] Hayt W.H. and Kemmerly J.E., *Engineering Circuit Analysis*, McGraw-Hill, 3rd ed. (1978).

[6] Kemeny J.G., Snell J.L., and Knapp A.W., *Denumerable Markov Chains*, Van Nostrand (1966).

Summing over z,

$$\sum_z U_z^{xy} = (1/2) \sum_z d(z) \left[R_{xy} + R_{yz} - R_{xz} \right]$$

But we have, $E[H_{xy}] = \sum_z U_z^{xy}$
proving the theorem.

Recall that the commute time result gives a precise characterization of the "sum" of expected transit times $(H_{xy} + H_{yx})$ between vertices x and y. While Theorem 5 can be used in giving yet another proof of the commute time result, we use it to derive a formula relating the "difference" of the expected transit times $(H_{yx} - H_{xy})$ between x and y.

Corollary 4

$$E[H_{yx}] - E[H_{xy}] = \sum_z d(z)(R_{xz} - R_{yz})$$

Proof. Obvious from Theorem 5!

We must mention here again that the analog of the above corollary, albeit in the random walk terminology, was proved in [4].

Thus the summary of this section of results is best expressed by

$$\boxed{\begin{aligned} E[H_{xy}] &= mR_{xy} + \frac{1}{2} \sum_z d(z) \left[R_{yz} - R_{xz} \right] \\ E[H_{yx}] &= mR_{xy} - \frac{1}{2} \sum_z d(z) \left[R_{yz} - R_{xz} \right] \end{aligned}}$$

3 General Networks

Let r_{xy} denote the resistance of branch xy of a general network. The underlying graph is now a *weighted* multigraph with *conductance* $c_{xy} = 1/r_{xy}$ as the weight on edge xy. And $c_{xz} = 0$ if x and z are non-adjacent. The weighted degree is $c(x) = \sum_{y \in N(x)} c_{xy}$, sum of the conductances of branches incident on node x. The transition probabilities p_{xz} of a random walk on G are now defined as follows.

$$\begin{aligned} Prob[\text{moving from x to z in one step}] = p_{xz} &= \frac{c_{xz}}{c(x)} \quad if\ \{x, z\} \in E, \\ &= 0 \qquad otherwise. \end{aligned}$$

It is not difficult to see that identity (*) is true in general networks provided we interpret the *weighted degree* $c(x)$ to be the degree of a vertex x. The proof of identity (**) below can be found in [2].

$$(**) \qquad V_{zy} = \frac{U_z^{zy}}{c(z)} \qquad \forall z$$

Using the above identity, it is straight forward to generalize all the results of the previous section. (i.e. just follow the same proofs.) Specifically, Theorem 3 holds true in a weighted sense yielding the following general version of Foster's theorem.

Triangle Inequality for Effective Resistances. The following triangle inequality for effective resistances is well known in electrical network theory: (also mentioned as Lemma 14 in [1])

$$R_{xz} + R_{zy} \geq R_{xy}$$

We show that the inequality can be made precise with the help of traversals in a random walk.

Corollary 3 *Any three nodes x, y, z of a network satisfy*

$$R_{xz} + R_{zy} - R_{xy} = \frac{U_x^{yz}}{d(x)} + \frac{U_y^{xz}}{d(y)}$$

Proof. By Theorem 2 and Corollary 2,

$$\begin{aligned} R_{xz} + R_{zy} - R_{xy} &= \Big(\frac{U_y^{xz}}{d(y)} + \frac{U_z^{yx}}{d(z)}\Big) + \Big(\frac{U_z^{xy}}{d(z)} + \frac{U_x^{yz}}{d(x)}\Big) - \Big(\frac{U_z^{xy}}{d(z)} + \frac{U_z^{yx}}{d(z)}\Big) \\ &= \frac{U_x^{yz}}{d(x)} + \frac{U_y^{xz}}{d(y)}. \\ \text{also} &= 2\frac{U_x^{yz}}{d(x)} = 2\frac{U_y^{xz}}{d(y)}. \end{aligned}$$

Notice that we have proved Corollaries 1 and 3 from first principles. Larry Ruzzo recently pointed out that Corollary 3 may be related to some already existing literature (Proposition 9-58 of [6]) on random walks.

Proposition 9-58.

$$\begin{aligned} E[H_{xy}] + E[H_{yz}] - E[H_{xz}] &= \frac{U_z^{xy}}{\pi(z)} \\ E[H_{xy}] + E[H_{yx}] &= \frac{U_x^{xy}}{\pi(x)} \end{aligned}$$

where $\pi(x) =$ stationary probability of $x = d(x)/2m$.
Surely enough, our interpretation of effective resistance (Theorems 1 and 2) together with Proposition 9-58 yields alternative proofs for Corollaries 1 and 3.

We now turn the above triangle inequality around deriving a result which deserves the status of a full theorem!

Theorem 5

$$E[H_{xy}] = \frac{1}{2}\sum_z d(z)\,[R_{xy} + R_{yz} - R_{xz}]$$

Proof. By the triangle inequality (Corollary 3) we have

$$R_{xy} + R_{yz} - R_{xz} = 2\frac{U_z^{xy}}{d(z)}$$

Turning it around,

$$U_z^{xy} = (1/2)d(z)\,[R_{xy} + R_{yz} - R_{xz}]$$

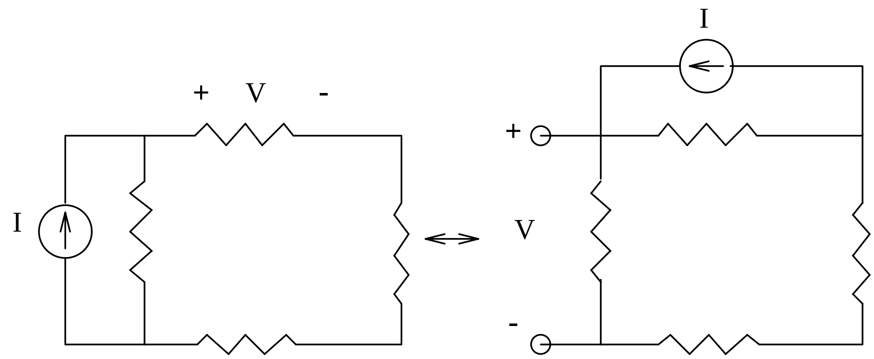

Figure 1: Reciprocity

In other words, the location of the current source and the resulting voltage may be interchanged without a change in voltage.

The theorem requires that the direction of the current source have the same correspondence with the polarity of the branch voltage in each position. (The theorem also has a dual in terms of voltage source and current measured in a branch.)

Recall that we have interpreted the voltage V_{wy} when a unit current flows into x and out of y to be $U_w^{xy}/d(w)$. So by the reciprocity theorem, we have

Corollary 2 *The expected number of traversals out of w along a specific edge during a random walk from x to y is the same as the expected number of traversals out of x along a specific edge in a random walk from w to y.*

We are now ready to prove Theorem 3.

Proof of Theorem 3: By Cor. 2 we have that

$$U_z^{xy}/d(z) = U_x^{zy}/d(x).$$

where the l.h.s. refers to a random walk from x to y, and the r.h.s. refers to a random walk from z to y. Taking sums on both sides,

$$\sum_{x \in N(y)} U_z^{xy}/d(z) = \sum_{x \in N(y)} U_x^{zy}/d(x) = 1 \text{ if } z \neq y$$

This is because the expected number of times we reach y from one of its neighbors in a *nontrivial* ($z \neq y$) random walk from z to y is precisely 1. Therefore

$$\sum_{y \in G} \sum_{x \in N(y)} U_z^{xy}/d(z) = n - 1$$

$$\text{i.e.} \quad \sum_{\{x,y\} \in E} (U_z^{xy}/d(z) + U_z^{yx}/d(z)) = n - 1.$$

we are done by Theorem 2.

Remark. From Theorem 3 it trivially follows that

$$\sum_{\{x,y\} \in E} E[C_{xy}] = 2m(n-1).$$

We are done by noting that $U_w/d(w)$ is the expected number of traversals out of w along a specific edge in our random roundtrip.

Interpretation of Polya's recurrence theorem. The relation between *recurrent* random walks on lattices and resistance of infinite networks is immediate from our characterization. A random walk is said to be *recurrent* if the walker is certain to return to the starting point. When a positive probability exists that the walker never returns, the random walk is called *transient.* The famous theorem of Polya asserts that *simple random walk on a d-dimensional lattice is recurrent for $d = 1, 2$ and transient for $d > 2$.* From Theorem 1 it is clear that the random walk is recurrent iff the effective resistance between the starting point and "point at infinity" goes to infinity! Intuitively, infinite resistance at the boundary always forces the *random walker* to return home! Doyle and Snell discuss [2] the relation between recurrence and effective resistance, by relating effective resistance to *escape probability*, the probability that the walker never returns to the starting point. The subtle difference in the interpretations can be explained. They consider effective resistance to be the reciprocal of the current flow when a 1 volt source is applied between the points of interest. Our characterization is based on the alternative definition of effective resistance, namely, the voltage developed between the points of interest when the source is 1 Amp current.

Let H_{xy} denote the *transit time* for the random walk from x to y, and let C_{xy} denote the *commute time* for the random roundtrip between x and y; i.e. $C_{xy} = H_{xy} + H_{yx}$. The following corollary gives a different way of proving one of the main results of [1].

Corollary 1 *Expected commute time $E[C_{xy}] = E[H_{xy}] + E[H_{yx}] = 2mR_{xy}$.*

Proof. Note that as long as we have not yet reached the "terminal" vertices x and y, we keep visiting other vertices of the graph. So by Theorems 1 and 2, we have

$$E[C_{xy}] = \sum_{w \in G} U_w = R_{xy} \sum_{w} d(w)$$

$$\text{i.e. } E[C_{xy}] = 2mR_{xy}.$$

It was pointed out to us by P.G. Doyle that the following theorem was originally due to R.M. Foster [3]. Foster proved a slightly more general theorem using network theory and we shall see in the next section that our proof can be extended to the general version.

Theorem 3

$$\sum_{\{x,y\} \in E} R_{xy} = n - 1.$$

Before proving this theorem, we interpret the Reciprocity Theorem of electrical networks in terms of random walks. The Reciprocity theorem is applicable only to single-source networks; we merely state the theorem, since the proof can be found in any standard circuit theory text book. (e.g. see [5] .)

Theorem 4 (Reciprocity) *The voltage V across any branch of a network, due to single current source I anywhere else in the network, will equal the voltage across the branch at which the source was originally located if the source is placed at the branch across which the voltage V was originally measured.*

Theorem 1 *The effective resistance R_{xy} between nodes x and y is exactly the expected number of traversals out of x along any specific edge (x,z) in a* simple *random walk starting at x and ending in y.*

Proof. Our characterization makes use of the well known analogy (see [2]) between random walks and electrical networks. Let U_z^{xy} be the expected number of visits to z (before reaching y) in a random walk from x to y. So $U_y^{xy}=0$, and considering all possible ways of reaching z, we have

$$\begin{aligned} U_z^{xy} &= \sum_w U_w^{xy} p_{wz} \\ &= \sum_w \frac{U_w^{xy}}{d(w)} \quad (z \neq x, y) \end{aligned}$$

The quantity of interest to us is

$$\frac{U_z^{xy}}{d(z)} = \sum_w \frac{1}{d(z)} \frac{U_w^{xy}}{d(w)} \qquad (z \neq x, y)$$

On the other hand, by Kirchoff's Current Law, it follows that

$$V_{zy} = \sum_w \frac{1}{d(z)} V_{wy} \qquad (z \neq x, y)$$

where V_{zy} is the voltage between z and y. Thus we get the following identity (*) by defining $V_{yy}=0$ and $V_{xy}=U_x^{xy}/d(x)$:

$$(*) \qquad\qquad V_{zy} = \frac{U_z^{zy}}{d(z)} \qquad\qquad \forall z$$

The right hand side is clearly the expected number of traversals out of x along a specific edge. And *effective resistance* R_{xy} is, by definition, the voltage V_{xy} when a unit current enters x and leaves y. Hence we will be done by proving that a unit current indeed enters x and leaves y. The current in any branch wz is

$$\begin{aligned} i_{wz} &= (V_{wy} - V_{zy})/1 \\ &= \frac{U_w^{xy}}{d(w)} - \frac{U_z^{xy}}{d(z)} \end{aligned}$$

i.e. i_{wz} is the expected number of *net traversals* along wz. This being the interpretation of current, the current entering x is $\sum_w i_{xz} = 1 = \sum_z i_{zy}$ equals the current leaving y.

Theorem 2 *R_{xy} is also the expected number of traversals out of w ($\neq x, y$) along any specific edge (w,z) in a* simple *random walk from x to y and then back to x.*

Proof. Let U_w denote the expected number of visits to w in a random "roundtrip": random walk from x to y and back to x. Clearly, $U_w = U_w^{xy} + U_w^{yx}$. Using the above identity (*) and the *superposition principle* we have

$$\begin{aligned} U_w &= V_{wy} d(w) - V_{wx} d(w) \\ &= V_{xy} d(w) \\ &= R_{xy} d(w) \end{aligned}$$

Random Walks and the Effective Resistance of Networks

Prasad Tetali

January 1990

Keywords: Electrical Networks, electrical impedance, random walks, markov chains

1 Introduction

A *simple* random walk on a graph is one in which from any vertex of the graph there is equal probability of moving to a neighboring vertex. Random walks in graphs have been found to have interesting analogies in resistor networks. Doyle and Snell made an excellent development of this topic in [2]. Given a graph, the underlying electrical network is the network obtained by replacing vertices by nodes and edges by electrical resistances. The *effective resistance* between any two nodes x and y can be defined as the voltage that develops between x and y when a unit current is maintained through them (i.e. enters one and leaves the other). We present here precise characterization of effective resistance in electrical networks in terms of random walks on underlying graphs. Our interpretation of effective resistance enables us in giving interesting new proofs for some known results. In particular, the interpretation of Polya's recurrence theorem in terms of electrical networks becomes obvious. We also obtain a precise version of the triangle inequality for effective resistances. This, in turn, is used in characterizing the *one-way transit times* between two vertices in terms of effective resistances of the network. (much in the spirit of the *commute time* result of [1]).

2 Simple Networks

By a *simple network* we mean a (finite) electrical network with all resistances equal. In this section we assume, with no loss of generality, we are dealing with unit resistances. We discuss the general case of networks with unequal resistances in the next section.

Consider a simple network with n nodes and m branches. Thus the underlying graph is a multigraph $G = (V, E)$ with n vertices and m edges. Let $d(x)$ denote the degree of vertex x. The transition probability p_{xz} of a simple random walk on G is defined as follows.

$$\begin{aligned} Prob[\text{moving from x to z in one step}] = p_{xz} &= \frac{1}{d(x)} \quad if\ \{x,z\} \in E, \\ &= 0 \qquad\quad otherwise. \end{aligned}$$

By a random walk *from x to y* we mean a random walk which begins at vertex x, goes around visiting vertices, and stops on reaching y.