Multidimensional Bin Packing and Other Related Problems: A Survey *

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Abstract

The bin packing problem is a well-studied problem in combinatorial optimization. In the classical bin packing problem, we are given a list of real numbers in $(0, 1]$ and the goal is to place them in a minimum number of *bins* so that no bin holds numbers summing to more than 1. The problem is extremely important in practice and finds numerous applications in scheduling, routing and resource allocation problems. Theoretically the problem has rich connections with discrepancy theory, iterative methods, entropy rounding and has led to the development of several algorithmic techniques. In this survey we consider several classical generalizations of bin packing problem such as geometric bin packing, vector bin packing and various other related problems.

In two-dimensional *geometric bin packing*, we are given a collection of rectangular items to be packed into a minimum number of unit size square bins. This variant has a lot of applications in cutting stock, vehicle loading, pallet packing, memory allocation and several other logistics and robotics related problems. In d-dimensional *vector bin packing*, each item is a d-dimensional vector that needs to be packed into unit vector bins. This problem is of great significance in resource constrained scheduling and in recent virtual machine placement in cloud computing. We also consider several other generalizations of bin packing such as geometric knapsack, strip packing and other related problems such as vector scheduling, vector covering etc. We survey algorithms for these problems in offline and online setting, and also mention results for several important special cases. We briefly mention related techniques used in the design and analysis of these algorithms and also survey some heuristics that work well in practice. In the end we conclude with a list of open problems.

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1 Introduction

The bin packing problem has been the corner stone of approximation algorithms and has been extensively studied starting from the early seventies. In the classical *bin packing problem*, we are given a list $I = \{i_1, i_2, \ldots, i_n\}$ of real numbers in the range $(0, 1]$, the goal is to place them in a minimum number of *bins* so that no bin holds numbers summing to more than 1.

Bin packing is a special case of the one-dimensional *cutting stock* problem [GG61], loading problem [EC71] and several scheduling related problems [CMM71]. In theoretical computer science, the bin packing problem was probably first studied by Garey, Graham and Ullman in 1972 [GGU72], from the standpoint of memory allocation problems such as table formatting, prepaging and file allocation. They noticed that finding a general placement algorithm for attaining the minimum number of bins appears to be impractical, and thus provided four heuristics: first fit (FF), best fit (BF), first fit decreasing height (FFDH) and best fit decreasing heights (BFDH). Soon Johnson, Demers, Ullman, Garey and Graham [JDU+74] published the first definitive analysis of the worst case guarantees of several bin packing approximation algorithms. The bin packing problem is well-known to be NP-hard [GJ79] and the seminal work of Johnson et al. initiated an extremely rich research area in approximation algorithms [Hoc96]. In fact the term *approximation algorithm* was coined by David S. Johnson [Joh73] in an influential and prescient paper in 1974 where he studied algorithms for bin packing and other packing and covering related optimization problems.

Bin packing is extremely useful in practice and has a lot of applications in various fields. Skiena [Ski99] has presented *market research* for the field of combinatorial optimization and algorithms, attempting to determine which algorithmic problems are most in demand for applications, by studying WWW traffic. Both bin packing and related knapsack problem were among top five most popular NP-hard problems. The implementations of bin packing and knapsack were the most needed among all NP-hard problems, even more than problems such as set-cover, traveling salesman and graph-coloring.

Garey and Johnson [GJ81], followed by Coffman et al. [CJGJ84], gave comprehensive surveys on bin packing algorithms. Coffman and Lueker also covered probabilistic analyses of packing algorithms in detail [CL91]. Galambos and Woeginger [GW95] gave an overview restricted mainly to online variants of bin packing problems. There had been many surveys on bin packing problems thereafter [Gon07, CGMV99, CW98]. The most recent, extensive coverage on 1-D bin packing was given by Coffman et al. [CJCG+13].

In this survey, we primarily focus on packing in higher dimensions due to its prominence in many real world applications. We primarily consider two generalizations of bin packing: GEOMETRIC BIN PACKING and VECTOR BIN PACKING.

In two-dimensional (2-D) *geometric bin packing* (GBP), we are given a collection of rectangular items to be packed into a minimum number of unit-size square bins. This variant and other higher dimensional GBP variants have vast applications in cutting stock, vehicle loading, pallet packing, memory allocation and several other logistics and robotics related problems [GG61, Ram92]. In two dimensions, packing objects into containers have many important applications, e.g., in the context of cutting out a given set of patterns from a given large piece of material minimizing waste, typically in sheet metal processing and apparel fabrication. In three dimensions, these problems are frequently encountered in minimizing storage space or container space for transportation. In this survey we consider the widely studied *orthogonal packing* case, where the items must be placed in the bin such that their sides are parallel to the sides of the bin. In any feasible solution, items are not allowed to overlap. Here two variants are usually studied, (i) where the items cannot be rotated (packing by *translations*), and (ii) they can be rotated by 90 degrees (packing by *restricted rigid* motions). These variants are also recurrent in practice, e.g., in apparel production usually there are patterns of weaving or texture on the material so that the position where a piece should be cut cannot be rotated arbitrarily.

In d-dimensional *vector bin packing* (VBP), each item is a d-dimensional vector that needs to be packed into unit vector bins. The problem is of great significance in resource constrained scheduling and appeared also recently in virtual machine placement in cloud computing [PTUW11]. For example, consider each job (item) has multiple resource requirements (dimensions) such as CPU, memory, I/O, disk, network etc. and each server (bin) has a bounded amount of these resources. The goal to assign all jobs to minimum number of servers, without violating the resource constraints, translates to the vector packing problem. Even in two dimensions, vector packing has many novel applications in layout design, logistics, loading and scheduling problems [SW84, Spi94].

These generalizations have been well studied since the 1970s. Baker, Coffman, and Rivest first considered orthogonal packings in two dimensions [BJR80]. At the same time Coffman et al. [JGJT80] gave performance bounds for level-oriented two-dimensional packing algorithms such as Next Fit Decreasing Height and First Fit Decreasing Height. Lodi, Martello and Monaci first gave a survey on two-dimensional packing problems [LMM02]. Epstein and van Stee gave a survey in [Gon07] on multi-dimensional bin packing. There has been consistent progress in the area since then. We will provide a detailed survey of related works in the later corresponding sections.

1.1 Organization of the Survey

In Section 2, we introduce related definitions and notation. In Section 3, we discuss results related to geometric bin packing. Thereafter in Section 4, we discuss results related to vector packing. Finally, in Section 5 we conclude with a list of open problems.

2 Preliminaries

In this section we introduce relevant notation and definitions required to define, analyze and classify bin packing related problems. Some more additional definitions will be introduced later on as required.

2.1 Approximation Algorithms and Inapproximability

Approximation Algorithm is an attempt to systematically measure, analyze, compare and improve the performance of heuristics for intractable problems. It gives theoretical insight on how to find fast solutions for practical problems, provides mathematical rigor to study and analyze heuristics, and also gives a metric for the difficulty of different discrete optimization problems.

Definition 2.1. (Approximation ratio) Given an algorithm $\mathcal{A}$ for a *minimization* problem Π, the (multiplicative) *approximation ratio* is:

$$\rho_{\mathcal{A}} = sup_{I \in \mathcal{I}} \left\{ \frac{\mathcal{A}(I)}{\mathsf{Opt}(I)} \right\},$$

where $\mathcal{A}(I)$ is the objective function value of the solution returned by algorithm $\mathcal{A}$ on instance $I \in \mathcal{I}$ and $\mathsf{Opt}(I)$ is the objective function value of the corresponding optimal solution.

In other words, an algorithm $\mathcal{A}$ for a minimization problem Π is called a ρ-approximation algorithm if $\mathcal{A}(I) \leq \rho \cdot \mathsf{Opt}(I)$ holds for every instance I of Π. An algorithm $\mathcal{A}$ for a maximization problem Π is called a ρ-approximation algorithm if $\mathcal{A}(I) \geq \frac{1}{\rho} \cdot \mathsf{Opt}(I)$ holds for every instance I of Π. This asymmetry ensures that $\rho \geq 1$ for all approximation algorithms.

In some cases, quality of the heuristic is measured in terms of additive approximation. In other words, an algorithm $\mathcal{A}$ for a minimization problem Π is called a σ-additive approximation algorithm if $\mathcal{A}(I) \leq \mathsf{Opt}(I) + \sigma$ holds for every instance I of Π. Additive approximation algorithms are relatively rare. Karmarkar-Karp's algorithm [KK82] for one-dimensional bin packing is one such example.

For detailed introduction to approximation algorithms, we refer the readers to the following books on approximation algorithms [Vaz01, WS11].

Definition 2.2. (Polynomial time approximation scheme (PTAS)) A problem is said to admit a *polynomial time approximation scheme* (PTAS) if for every constant $\epsilon > 0$, there is a *poly*(n)-time algorithm with approximation ratio $(1 + \epsilon)$ where n is the size of the input. Here running time can be as bad as $O(n^{f(1/\epsilon})$ for any function f that depends only on ϵ.

If the running time of PTAS is $O(f(1/\epsilon) \cdot n^c)$ for some function f and a constant c that is independent of ϵ, we call it to be an *efficient polynomial time approximation scheme* (EPTAS).

On the other hand, if the running time of PTAS is polynomial in both n and $1/\epsilon$, it is said to be a *fully polynomial time approximation scheme* (FPTAS).

Assuming $\mathsf{P} \neq \mathsf{NP}$, a PTAS is the best result we can obtain for a strongly NP-hard problem. Already in the 1D case, a simple reduction from the PARTITION problem shows that it is NP-hard to determine whether a set of items can be packed into two bins or not, implying that no approximation better than $3/2$ is possible. However, this does not rule out the possibility of an $\mathsf{Opt} + 1$ guarantee where Opt is the number of bins required in the optimal packing. Hence it is insightful to consider the *asymptotic approximation ratio.*

Definition 2.3. (Asymptotic approximation ratio (AAR)) The *asymptotic approximation ratio* of an algorithm $\mathcal{A}$ is ρ if the output of the algorithm has objective function value at most $\rho \cdot \mathsf{Opt}(I) + \delta$ for some constant δ, for each instance I.

In this context the approximation ratio defined as in Definition 2.1, is also called to be the *(absolute) approximation ratio.* If $\delta = 0$, then $\mathcal{A}$ has (absolute) approximation guarantee ρ.

Definition 2.4. (Asymptotic PTAS (APTAS)) A problem is said to admit an *asymptotic polynomial time approximation scheme* (APTAS) if for every $\epsilon > 0$, there is a poly-time algorithm with asymptotic approximation ratio of $(1 + \epsilon)$.

If the running time of APTAS is polynomial in both n and $1/\epsilon$, it is said to be an *asymptotic fully polynomial time approximation scheme* (AFPTAS).

Note that NP optimization problems whose decision versions are all polynomial time reducible to each other (due to NP-completeness), can behave very differently in their approximability. For example classical bin packing problem admits an APTAS, whereas no polynomial factor approximation is known for the traveling salesman problem. This anomaly is due to the fact that reductions between NP-complete problems preserve polynomial time computability, but not the quality of the approximate solution.

PTAS is the class of problems that admit polynomial time approximation scheme. On the other hand, APX is the class of problems that have a constant-factor approximation. Clearly $\mathsf{PTAS} \subseteq \mathsf{APX}$. In fact the containment is strict unless $\mathsf{P} = \mathsf{NP}$. A problem is APX-hard if there is a constant $\delta > 0$ such that the problem does not admit a $(1+\delta)$ approximation unless $\mathsf{P} = \mathsf{NP}$.

Theorem 2.5. [CP91] If a problem $\mathscr{F}$ is APX-hard then it does not admit a PTAS unless $\mathsf{P} = \mathsf{NP}$.

In fact there is no quasi-polynomial time approximation scheme (QPTAS) for any APX-hard problem unless $\mathsf{NP} \subseteq \mathsf{QP}$.

Online Algorithms: Optimization problems where the input is received in an online manner and the output must be produced online are called online problems. Bin packing is also one of the key problems in online algorithms. Let us define the notion of a competitive ratio which will be useful when we discuss some related results in online algorithms in later sections.

Definition 2.6. (Competitive Ratio) Given an online algorithm $\mathcal{A}$ for a *minimization* problem Π, the *competitive ratio* is:

$$\rho_{\mathcal{A}} = sup_{I \in \mathcal{I}} \left\{ \frac{\mathcal{A}(I)}{\mathsf{Opt}(I)} \right\},$$

where $\mathcal{A}(I)$ is the objective function value of the solution returned by the online algorithm $\mathcal{A}$ on instance I over the set of all finite input sequences $\mathcal{I}$ and $\mathsf{Opt}(I)$ is the objective function value of the corresponding optimal offline solution.

Asymptotic competitive ratio and *absolute competitive ratio* are defined analogously. In general, there are no requirements or assumptions concerning the computational efficiency of an online algorithm. However, in practice, we usually seek polynomial time online algorithms.

There are few others metrics to measure the quality of a packing, such as *random-order ratio* [Ken96], *accommodation function* [BLN01], *relative worst-order ratio* [BF07], *differential approximation measure* [DGP98] etc.

2.2 One Dimensional Bin Packing

Before going to multidimensional bin packing, we give a brief description of the results in 1-D bin packing. Here we focus primarily on very recent results. For a detailed survey and earlier results we refer the interested reader to [CJCG$^+$13].

2.2.1 Offline 1-D Bin Packing

The earliest algorithms for one dimensional (1-D) bin packing were simple greedy algorithms such as First Fit (FF), Next Fit (NF), First Fit Decreasing Heights (FFDH), Next Fit Decreasing Heights (NFDH) etc. In their celebrated work, de la Vega and Lueker [dlVL81] gave the first APTAS by introducing linear grouping that reduces the number of different item types. Algorithms based on other item grouping or rounding based techniques have been used in many related problems. The result was substantially improved by Karmarkar and Karp [KK82] who gave a guarantee of $\mathsf{Opt} + O(\log^2 \mathsf{Opt})$ by providing an iterative rounding for a linear programming formulation. It was then improved by Rothvoß [Rot13] to $\mathsf{Opt} + O(\log \mathsf{Opt} \cdot \log\log \mathsf{Opt})$ using ideas from discrepancy theory. Very recently, Hoberg and Rothvoß [HR15] achieved approximation ratio of $\mathsf{Opt} + O(\log \mathsf{Opt})$ using discrepancy method coupled with a novel 2-stage packing approach. On the other hand, the possibility of an algorithm with an $\mathsf{Opt}+1$ guarantee is still open. This is one of the top ten open problems in the field of approximation algorithms mentioned in [WS11].

Table 1 summarizes different algorithms and their performance guarantees. Here $T_\infty \approx 1.69$ is the well-known *harmonic* constant that appears ubiquitously in the context of bin packing.

The *Gilmore-Gomory LP relaxation* [GG61] is used in [dlVL81, KK82, Rot13] to obtain better approximation. This LP is of the following form:

$$min\{\mathbf{1}^T x | Ax = \mathbf{1}, x \geq 0\} \tag{1}$$

Here A is the pattern matrix that consists of all column vectors $\{p \in \mathbb{N}^n | p^T s \leq \mathbf{1}\}$ where $s := (s_1, s_2, \ldots, s_n)$ is the size vector for the items. Each such column p is called a pattern and corresponds to a feasible multiset of items that can be assigned to a single bin. Now if we only consider patterns $p \in \{0,1\}^n$, LP (1) can be interpreted as an LP relaxation of a set cover problem, in which a set I of items has to be covered by *configurations* from the collection $\mathcal{C} \subseteq 2^I$, where each configuration $C \in \mathcal{C}$ corresponds to a set of items that can be packed into a

Table 1: Approximation algorithms for one dimensional bin packing

Algorithm	Performance Guarantee	Techniques
Next Fit [JDU+74]	$2 \cdot \mathsf{Opt}$	Greedy, Online,
Next Fit Decreasing [JDU+74]	$T_\infty \cdot \mathsf{Opt} + O(1)$ [BC81]	Presorting
First Fit [JDU+74]	$\lfloor 1.7\mathsf{Opt} \rfloor$ [DS13]	Greedy, Online
First Fit Decreasing[JDU+74]	$\frac{11}{9}\mathsf{Opt} + \frac{6}{9}$ [DLHT13]	Presorting
de la Vega and Lueker [dlVL81]	$(1+\epsilon)\mathsf{Opt} + O(\frac{1}{\epsilon^2})$	Linear grouping
Karp and Karmarkar [KK82]	$\mathsf{Opt} + O(\log^2 \mathsf{Opt})$	Iterative rounding
Rothvoß [Rot13]	$\mathsf{Opt} + O(\log \mathsf{Opt} \cdot \log\log \mathsf{Opt})$	Discrepancy methods
Hoberg and Rothvoß [HR15]	$\mathsf{Opt} + O(\log \mathsf{Opt})$	Discrepancy methods

bin:

$$min\left\{ \sum_{C \in \mathcal{C}} x_C : \sum_{C \ni i} x_C \geq 1 \quad (i \in I), x_C \in \{0,1\} \quad (C \in \mathcal{C}) \right\}. \tag{2}$$

This configuration LP is also used in other algorithms for multidimensional bin packing and we will continue the discussion of configuration LPs in later sections.

Let Opt and Opt_f be the value of the optimal integer solution and fractional solution for LP (1) respectively. Although LP (1) has an exponential number of variables, one can compute a basic solution x with $\mathbf{1}^T x \leq \mathsf{Opt}_f + \delta$ in time polynomial in n and $1/\delta$ using the Grötschel -Lovász-Schrijver variant of the Ellipsoid method [GLS88] or the Plotkin-Shmoys-Tardos framework [PST95, AHK12]. In fact the analysis of [HR15], only shows an upper bound of $O(\log \mathsf{Opt})$ on the additive integrality gap of LP (1). It has been conjectured in [ST97] that the LP has the *Modified Integer Roundup Property*, i.e., $\mathsf{Opt} \leq \lceil \mathsf{Opt}_f \rceil + 1$. The conjecture has been proved true for the case when the instance contains at most 7 different item sizes [SS09]. Recently, Eisenbrand et al. [EPR13] found a connection between coloring permutations and bin packing, that shows that *Beck's Three Permutation Conjecture* (any three permutations can be bi-colored with O(1) discrepancy) would imply a $O(1)$ integrality gap for instances with all items sizes bigger than $1/4$. However, Newman, Neiman, and Nikolov [NNN12] found a counterexample to Beck's conjecture. Using these insights Eisenbrand et al. [EPR13] showed that a broad class of algorithms cannot give an $o(\log n)$ gap. Rothvoß [Rot12] further explored the connection to discrepancy theory and gave a rounding using Beck's entropy method achieving an $O(\log^2 \mathsf{Opt})$ gap alternatively. The later improvement to $O(\log \mathsf{Opt})$ in [Rot13, HR15] arose from the constructive partial coloring lemma [LM12] and gluing techniques. Recently Goemans and Rothvoß [GR14] also have shown polynomiality for bin packing when there are $O(1)$ number of item types.

Bin packing problem is also well-studied when the number of bins is some fixed constant k. If the sizes of the items are polynomially bounded integers, then the problem can be solved *exactly* using dynamic programming in $n^{O(k)}$ time, for an input of length n. Along with APTAS for bin packing, this implies a QPTAS for bin packing, significantly better than $3/2$, the hardness of (absolute) approximation for the problem. However, Jansen et al. [JKMS13] showed *unary bin packing* (where item sizes are given in unary encoding) is $\mathsf{W}[1]$-hard and thus the running time for fixed number of bins k, cannot be improved to $f(k) \cdot n^{O(1)}$ for any function of k, under standard complexity assumptions.

2.2.2 Online 1-D Bin Packing

An online bin packing algorithm uses *k-bounded space* if, for each item, the choice of where to pack it, is restricted to a set of at most k open bins. Lee and Lee [LL85] gave a $O(1)$-*bounded space* algorithm that achieve asymptotic competitive ratio of $T_\infty \approx 1.69$. They also established tightness. Seiden [Sei02] gave a new algorithm, HARMONIC++, whose asymptotic performance ratio is at most 1.58889. Ramanan et al. [RBLL89] showed that Harmonic-type algorithms cannot achieve better than 1.58333 asymptotic competitive ratio. Very recently, this Harmonic lower bound was beaten by Heydrich and van Stee [HvS15] who presented an online algorithm with asymptotic performance ratio of at most 1.5817 using a new type of *interval classification*. They also gave a lower bound of 1.5766 for any interval classification algorithm. In general the best known lower bound for asymptotic competitive ratio is 1.54014 [vV92]. Very recently, Balogh et al. [BBD+15] presented an online bin packing algorithm with an *absolute* competitive ratio of 5/3 which is optimal.

Online bin packing has also been studied in probabilistic settings. Shor [Sho91] gave tight-bounds for average-case online bin packing. Other related algorithms for online stochastic bin packing are *Sum of Squares* algorithm by Csirik et al. [CJK+06] and primal-dual based algorithms in [GR12].

2.3 Multidimensional Bin Packing

We will now discuss preliminaries related to multidimensional bin packing. We will consider the offline setting, where all items are known a priori. We also briefly survey results in the online setting, when the items appear one at a time and we need to decide packing an item without knowing the future items.

2.3.1 Geometric Packing

Definition 2.7. (Two-Dimensional Geometric Bin Packing (2-D GBP)) In two-dimensional *geometric bin packing* (2-D GBP), we are given a collection of n rectangular items $I := \{r_1, r_2, \ldots, r_n\}$ where each rectangle r_k is specified by its width and height (w_k, h_k) such that w_k, h_k are rational numbers in $[0, 1]$. The goal is to pack all rectangles into a minimum number of unit square bins.

We consider the widely studied *orthogonal packing* case, where the items must be placed in the bin such that their sides are parallel to the sides of the bin. In any feasible solution, items are not allowed to overlap. Here two variants are usually studied, (i) where the items cannot be rotated, and (ii) they can be rotated by 90 degrees.

We will also mention some results related to strip packing and geometric knapsack problems, two other geometric generalizations of bin packing, in Section 3.

Definition 2.8. (Strip Packing (2-D SP)) In two-dimensional *strip packing* (2-D SP), we are given a strip of unit width and infinite height, and a collection of n rectangular items $I := \{r_1, r_2, \ldots, r_n\}$ where each rectangle r_k is specified by its width and height (w_k, h_k) such that w_k, h_k are rational numbers in $[0, 1]$. The goal is to pack all rectangles into the strip minimizing the height.

This is a variant of cutting stock problem, well studied in optimization.

Definition 2.9. (Geometric Knapsack (2-D GK)) In two-dimensional *geometric knapsack* (2-D GK), we are given a unit square bin and a collection of two dimensional rectangles $I := \{r_1, r_2, \ldots, r_n\}$ where each rectangle r_k is specified by its width and height (w_k, h_k) and profit p_k such that w_k, h_k, p_k are rational numbers in $[0, 1]$. The goal is to find the maximum profit subset that can be feasibly packed into the bin.

Multidimensional variants of above three geometric problems are defined analogously using d-dimensional rectangular parallelepipeds (also known as d-orthotope, the generalization of rectangles in higher dimensions) and d-dimensional cuboids (also known as d-cube, the generalization of squares in higher dimensions). We will discuss 3-dimensional variants in more detail in Section 3.

2.3.2 Vector Packing

Now we define vector bin packing, the nongeometric generalization of bin packing.

Definition 2.10. (Vector Bin Packing (d-D VBP)) In d-dimensional *vector packing* (d-D VBP), we are given a set of n rational vectors $I := \{v_1, v_2, \ldots, v_n\}$ from $[0,1]^d$. The goal is to partition them into sets (bins) $B_1, B_2, \ldots, B_m$ such that $||\sigma_{B_j}||_\infty \leq 1$ for $1 \leq j \leq m$ where $\sigma_{B_j} = \sum_{v_i \in B_j} v_i$ is the sum of vectors in B_j, and we want to minimize m, the number of bins.

In other words, the goal is to pack all the vectors into minimum number of bins so that for every bin the sum of packed vectors in the bin should not exceed the vector of the bin in each dimension.

We now define related vector scheduling and vector bin covering problems.

Definition 2.11. (Vector Scheduling (d-D VS)) In d-dimensional *vector scheduling* (d-D VS), we are given a set of n rational vectors $I := \{v_1, v_2, \ldots, v_n\}$ from $[0,1]^d$ and an integer m. The goal is to partition I into m sets $B_1, B_2, \ldots, B_m$ such that $max_{1 \leq i \leq m} ||\sigma_{B_i}||_\infty$ is minimized, where $\sigma_{B_i} = \sum_{v_i \in B_i} v_i$ is the sum of vectors in B_i.

For $d = 1$, this just reduces the classical multiprocessor scheduling.

Definition 2.12. (Vector Bin Covering (d-D VBC)) In d-dimensional *vector bin covering* (d-D VBC), we are given a set of n rational vectors $I := \{v_1, v_2, \ldots, v_n\}$ from $[0,1]^d$. The goal is to partition them into sets (bins) $B_1, B_2, \ldots, B_m$ such that $\sigma_{B_j} \geq 1$ in all dimensions for all $j \in [m]$, where $\sigma_{B_j} = \sum_{v_i \in B_j} v_i$ is the sum of vectors in B_1, and we want to maximize m, the number of bins.

For $d = 1$, classical bin covering problem admits APTAS [JSO03].

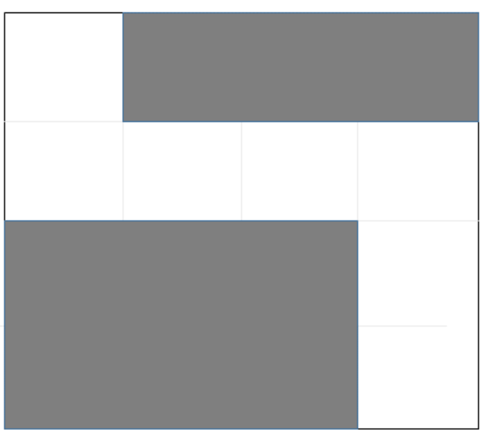

Figure 1: Two rectangles of size $\frac{1}{2} \times \frac{3}{4}$ and $\frac{1}{2} \times \frac{3}{4}$ can be packed into one bin

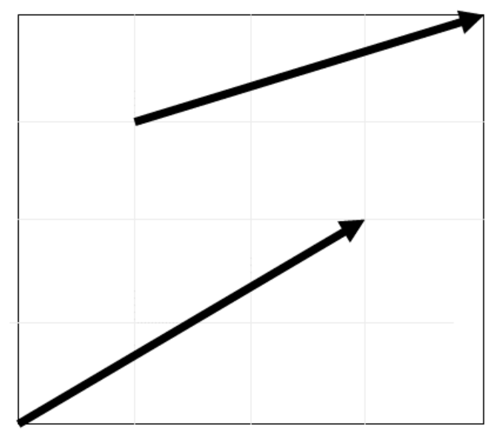

Figure 2: Two vectors $(\frac{1}{2}, \frac{3}{4})$ and $(\frac{1}{4}, \frac{3}{4})$ cannot be packed into one vector bin as their sum exceeds one in the second dimension

2.3.3 Relation between the problems

Figure 1 and 2 show the difference between geometric packing and vector packing. Given a set of vectors, one can easily determine whether they can be packed into one unit bin by just checking whether the sum along each coordinate is at most one or not. However for geometric bin packing, it is NP-hard to determine whether a set of rectangles can be packed into one unit square bin or not, implying that no (absolute) approximation better than 2 is possible even for 2-D GBP.

Note that both geometric knapsack and strip packing are closely related to geometric bin packing. Results and techniques related to strip packing and knapsack have played a major role in improving the approximation for geometric bin packing. If all items have same height then d-dimensional strip packing reduces to $(d-1)$-dimensional geometric bin packing. On the other hand to decide whether a set of rectangles (w_i, h_i) for $i \in [n]$ can be packed into m bins, one can define a 3-D geometric knapsack instance with n items $(w_i, h_i, 1/m)$ and profit $(w_i \cdot h_i \cdot 1/m)$ and decide if there is a feasible packing with profit $\sum_{i \in n}(w_i \cdot h_i \cdot 1/m)$. Figure 3 shows the relation between different generalizations of bin packing.

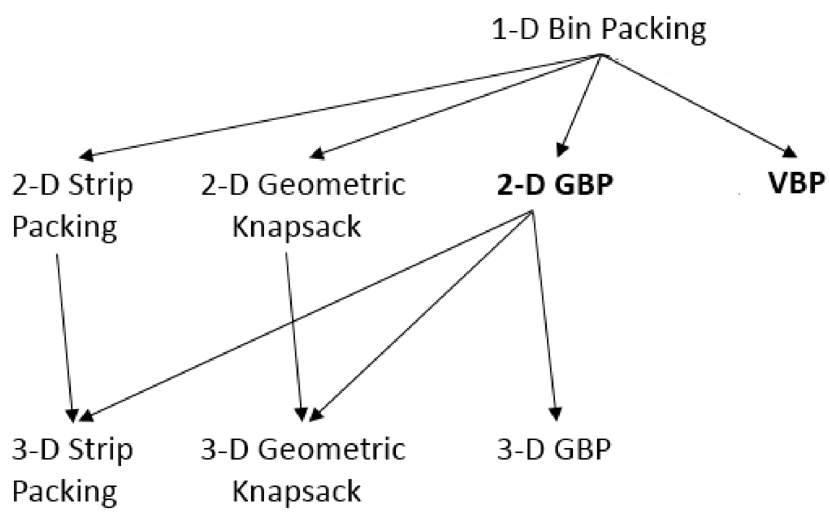

Figure 3: Generalizations of bin packing problems

There are few other generalizations of bin packing such as weighted bipartite edge coloring. We do not cover them here and we refer the readers to [Kha15, KS15].

2.4 Techniques:

Now we describe some techniques, heavily encountered in multidimensional packing.

2.4.1 Next Fit Decreasing Height (NFDH)

The Next Fit Decreasing Height (NFDH) procedure was introduced by Coffman et al. [JGJT80] for 2-D packing. NFDH considers items in a non-increasing order of height and greedily assigns items in this order into *shelves*, where a *shelf* is a row of items having their bases on a line that is either the base of the bin or the line drawn at the top of the highest item packed in the shelf below. More specifically, items are packed left-justified starting from bottom-left corner of the bin, until the next item cannot be included. Then the shelf is closed and the next item is used to define a new shelf whose base touches the tallest (left most) item of the previous shelf. If the shelf does not fit into the bin, the bin is closed and a new bin is opened. The procedure continues until all the items are packed. This simple heuristic works quite well for small items. Some key properties of NFDH are following:

Lemma 2.13. [MM68] Let B be a rectangular region with width w and height h. We can pack small rectangles (with both width and height less than ϵ) with total area A using NFDH into B if $w \geq \epsilon$ and $w \cdot h \geq 2A + w^2/8$.

Lemma 2.14. [CLM05] Given a set of items of total area of V and each having height at most one, they can be packed in at most $4V + 3$ bins by NFDH.

Lemma 2.15. [JGJT80] Let B be a rectangular region with width w and height h. If we pack small rectangles (with both width and height less than ϵ) using NFDH into B, total $w \cdot h - (w + h) \cdot \epsilon$ area can be packed, i.e., the total wasted volume in B is at most $(w + h) \cdot \epsilon$.

In fact it can be generalized to d-dimensions.

Lemma 2.16. [BCKS06] Let C be a set of d-dimensional cubes (where $d \geq 2$) of sides smaller than ϵ. Consider NFDH heuristic applied to C. If NFDH cannot place any other cube in a rectangle R of size $r_1 \times r_2 \times \cdots r_d$ (with $r_i \leq 1$), the total wasted (unfilled) volume in that bin is at most: $\epsilon \sum_{i=1}^{d} r_i$.

2.4.2 Configuration LP

The best known approximations for most bin packing type problems are based on strong LP formulations called configuration LPs. Here, there is a variable for each possible way of feasibly packing a bin (called a *configuration*). This allows the packing problem to be cast as a set covering problem, where each item in the instance I must be covered by some configuration. Let $\mathcal{C}$ denote the set of all valid configurations for the instance I. The configuration LP is defined as:

$$\min \left\{ \sum_{C \in \mathcal{C}} x_C : \qquad \sum_{C \ni i} x_C \geq 1 \quad \forall i \in I, \qquad x_C \geq 0 \quad \forall C \in \mathcal{C} \right\}. \tag{3}$$

As the size of $\mathcal{C}$ can possibly be exponential in the size of I, one typically considers the dual of the LP given by:

$$\max \left\{ \sum_{i \in I} v_i : \qquad \sum_{i \in C} v_i \leq 1 \quad \forall C \in \mathcal{C}, \qquad v_i \geq 0 \quad \forall i \in I \right\}. \tag{4}$$

The separation problem for the dual is the following knapsack problem. Given set of weights v_i, is there a feasible configuration with total weight of items more than 1. From the well-known connection between separation and optimization [GKPV01, PST95, GLS88], solving the dual separation problem to within a $(1 + \epsilon)$ accuracy suffices to solve the configuration LP within $(1 + \epsilon)$ accuracy. We refer the readers to [Rot12] for an explicit proof that, for any set family $\mathcal{C} \subseteq 2^{[n]}$, if the dual separation problem can be approximated to $(1 + \epsilon)$-factor in time $T(n, \epsilon)$ then the corresponding column-based LP can be solved within an arbitrarily small additive error δ in time $poly(n, \frac{1}{\delta}) \cdot T(n, \Omega(\frac{\delta}{n}))$. This error term cannot be avoided as otherwise we can decide the PARTITION problem in polynomial time. For 1-D BP, the dual separation problem admits an FPTAS, i.e., it can be solved in time $T(n, \epsilon) = poly(n, \frac{1}{\epsilon})$. Thus the configuration LP can be solved within arbitrarily small additive constant error δ in time $poly(n, \frac{1}{\delta}) \cdot poly(n, O(\frac{n}{\delta}))$. For multidimensional bin packing, the dual separation problem admits a PTAS, i.e., can be solved in time $T(n, \epsilon) = O(n^{f(\frac{1}{\epsilon})})$. Thus the configuration LP can be solved within $(1 + \delta)$ multiplicative factor in time $poly(n, \frac{1}{\delta}) \cdot T(n, \Omega(\frac{\delta n}{n}))$, i.e., in time $O(n^{O(f(\frac{1}{\delta}))})$.

Note that the configurations in (3) are defined based on the original item sizes (without any rounding). However, for more complex problems (say 3-D GBP) one cannot hope to solve such an LP to within $(1+\epsilon)$ (multiplicative) accuracy, as the dual separation problem becomes at least as hard as 2-D GBP. In general, given a problem instance I, one can define a configuration LP in multiple ways (say where the configurations are based on rounded sizes of items in I, which might be necessary if the LP with original sizes is intractable). For the special case of 2-D GBP, the separation problem for the dual (4) is the 2-D geometric knapsack problem for which the best known result is only a 2-approximation. However, Bansal et al. [BCJ+09] showed that the configuration LP (3) with original sizes can still be solved to within $1+\epsilon$ accuracy (this is a non-trivial result and requires various ideas).

Similarly for the case of vector bin packing, the separation problem for the dual (4) is the vector knapsack problem which can be solved to within $1+\epsilon$ accuracy [FC84]. However, there is no EPTAS even for 2-D vector knapsack [KS10].

The fact that solving the configuration LP does not incur any loss for 2-D GBP or VBP plays a key role in the present best approximation algorithms.

2.4.3 Algorithms based on rounding items to constant number of types

Rounding of items to $O(1)$ types has been used either implicitly [BLS05] or explicitly [dlVL81, KK82, Cap02, JP13, KO07], in almost all bin packing algorithms to reduce the problem complexity and make it tractable. Let I be a given set of items and s_x be the size of item $x \in I$. Define a partial order on bin packing instances as follows: $I \leq J$ if there exists a bijective function $f : I \to J$ such that $s_x \leq s_{f(x)}$ for each item $s \in I$. J is then called a rounded up instance of I. One of the key properties of rounding items is as follows:

Lemma 2.17. [KK82] $I \leq J$ implies $\mathsf{Opt}(I) \leq \mathsf{Opt}(J)$.

There are typically two types of rounding: either the size of an item in some coordinate (such as width or height) is rounded in an instance-oblivious way (e.g., in harmonic rounding [LL85, Cap02], or rounding sizes to geometric powers [KK82]), or it is rounded in an input sensitive way (e.g., in linear grouping [dlVL81]).

Linear grouping: Linear grouping was introduced by Fernandez de la Vega and Lueker [dlVL81] in the first approximation scheme for 1-D bin packing problem. It is a technique to reduce the number of distinct item sizes. The scheme works as follows, and is based on a parameter k, to be fixed later. Sort the n items by nonincreasing size and partition them into $\lceil 1/k \rceil$ groups such that the first group consists of the largest $\lceil nk \rceil$ pieces, next group consists of the next $\lceil nk \rceil$ largest items and so on, until all items have been placed in a group. Apart from the last group all other groups contain $\lceil nk \rceil$ items and the last group contains $\leq nk$ items. The rounded instance $\tilde{I}$ is created by discarding the first group, and for each other group, the size of an item is rounded up to the size of the largest item in that group. The following lemma shows that the optimal packing of these rounded items is very close to the optimal packing of the original items.

Lemma 2.18. [dlVL81] Let $\tilde{I}$ be the set of items obtained from an input I by applying linear grouping with group size $\lceil nk \rceil$, then

$$\mathsf{Opt}(\tilde{I}) \leq \mathsf{Opt}(I) \leq \mathsf{Opt}(\tilde{I}) + \lceil nk \rceil$$

and furthermore, any packing of $\tilde{I}$ can be used to generate a packing of I with at most $\lceil nk \rceil$ additional bins.

If all items are $> \epsilon$, then $n\epsilon < \mathsf{Opt}$. So, for $k = \epsilon^2$ we get that any packing of $\tilde{I}$ can be used to generate a packing of I with at most $\lceil n\epsilon^2 \rceil \leq n\epsilon^2 + 1 < \epsilon \cdot \mathsf{Opt} + 1$ additional bins. See [Kha15] for a slightly modified version of linear grouping that does not loose the additive constant of 1.

Geometric grouping: Karmarkar and Karp [KK82] introduced a refinement of linear grouping called geometric grouping with parameter k. Let $\alpha(I)$ be the smallest item size of an instance I. For $r = 0, 1, \ldots, \lfloor \log_2 \frac{1}{\alpha(I)} \rfloor$, let I_r be the instance consisting of items $i \in I$ such that $s_i \in (2^{-(r+1)}, 2^{-r}]$. Let J_r be the instances obtained by applying linear grouping with parameter $k \cdot 2^r$ to I_r. If $J = \cup_r J_r$ then:

Lemma 2.19. [KK82] $\mathsf{Opt}(J) \le \mathsf{Opt}(I) \le \mathsf{Opt}(J) + k\lceil \log_2 \frac{1}{\alpha(I)} \rceil$.

Harmonic rounding: Lee and Lee [LL85] introduced a harmonic algorithm (*harmonic*$_k$) for online 1-D bin packing, where each item j with $s_j \in (\frac{1}{q+1}, \frac{1}{q}]$ for $q \in \{1, 2, \cdots, (k-1)\}$, is rounded to $1/q$. Then q items of type $1/q$ can be packed together in a bin. So for each type q, we open one bin B_q and only items of type q are packed into that bin and a new bin for type q items is opened when B_q is full. Let $t_1 = 1$, $t_{q+1} := t_q(t_q + 1)$ for $q \ge 1$. Let $m(k)$ be the integer with $t_{m(k)} < k < t_{m(k)+1}$. It is shown in [LL85] that the asymptotic approximation ratio of *harmonic*$_k$ is $T_k = \sum_{q=1}^{m(k)} \frac{1}{t_q} + \frac{1}{t_{m(k)+1}} \cdot \frac{k}{k-1}$. When $k \to \infty$, $T_\infty = 1.691..$, this is the harmonic constant, ubiquitous in bin packing. Caprara [Cap02] introduced the *harmonic decreasing height* algorithm for 2-D GBP with asymptotic approximation ratio of T_∞, where widths are rounded according to harmonic rounding and then same width items are packed using NFDH. We refer the readers to [Eps15] for more related applications of the Harmonic algorithm in online and bounded space algorithms.

2.4.4 Round and Approx (R&A) Framework

Now we describe the R&A Framework as described in [BCS09]. It is the key framework used to obtain present best approximation algorithms for 2-D geometric bin packing and vector bin packing.

1. Solve the LP relaxation of (3) using the APTAS ([BCJ$^+$09] for 2-D GBP, [FC84] for VBP). Let x^* be the (near)-optimal solution of the LP relaxation and let $z^* = \sum_{C \in \mathcal{C}} x^*_C$. Let r the number of configurations in the support of x^*.
2. Initialize a $|\mathcal{C}|$-dimensional binary vector x^r to be an all-0 vector. For $\lceil (\ln \rho) z^* \rceil$ iterations repeat the following: select a configuration $C' \in \mathcal{C}$ at random with probability $x^*_{C'}/z^*$ and let $x^r_{C'} := 1$.
3. Let S be the remaining set of items not covered by x^r, i.e., $i \in S$ if and only if $\sum_{C \ni i} x^r_C = 0$. On set S, apply the ρ-approximation algorithm $\mathcal{A}$ that rounds the items to $O(1)$ types and then pack. Let x^a be the solution returned by $\mathcal{A}$ for the residual instance S.
4. Return $x = x^r + x^a$.

Let $\mathsf{Opt}(S)$ and $\mathcal{A}(S)$ denote the value of the optimal solution and the approximation algorithm used to solve the residual instance, respectively. Since the algorithm uses randomized rounding in step 2, the residual instance S is not known in advance. However, the algorithm should perform "well" independent of S. For this purpose, Bansal, Caprara and Sviridenko [BCS09] defined the notion of *subset-obliviousness* where the quality of the approximation algorithm to solve the residual instance is expressed using a small collection of vectors in $\mathbb{R}^{|I|}$.

Definition 2.20. An asymptotic ρ-approximation for the set covering problem defined in (1), is called subset-oblivious if, for any fixed $\epsilon > 0$, there exist constants k, Λ, β (possibly dependent on ϵ), such that for every instance I of (1), there exist vectors $v^1, v^2, \cdots, v^k \in \mathbb{R}^{|I|}$ that satisfy the following properties:

1. $\sum_{i \in C} v_i^j \le \Lambda$, for each configuration $C \in \mathcal{C}$ and $j = 1, 2, \cdots, k$;
2. $\mathsf{Opt}(I) \ge \sum_{i \in I} v_i^j$ for $j = 1, 2, \cdots, k$;
3. $\mathcal{A}(S) \le \rho \sum_{i \in S} v_i^j + \epsilon \mathsf{Opt}(I) + \beta$, for each $S \subseteq I$ and $j = 1, 2, \cdots, k$.

Roughly speaking, the vectors are analogues to the sizes of items and are introduced to use the properties of the dual of (1). Property 1 says that the vectors divided by constant Λ must be feasible for (2). Property 2 provides lower bound for $OPT(I)$ and Property 3 guarantees that the $\mathcal{A}(S)$ is not significantly larger than ρ times the lower bound in Property 2 associated with S.

The main result about the R&A is the following.

Theorem 2.21. (simplified) If a problem has a asymptotic ρ-approximation algorithm that is subset oblivious, and the configuration LP with original item sizes can be solved to within $(1+\epsilon)$ accuracy in polynomial time for any $\epsilon > 0$, then the R&A framework gives a $(1+\ln \rho)$-asymptotic approximation.

Very recently, Bansal and Khan [BK14] extended the R&A framework to any constant rounding based algorithms for 2-D GBP. Then Bansal, Elias and Khan [BEK16] further showed that any constant rounding based algorithm for VBP is also subset-oblivious.

One can derandomize the above procedure and get a deterministic version of R&A method in which Step 2 is replaced by a greedy procedure that defines x^r guided by a suitable potential function. See [BCS09] for the details of derandomization.

3 Geometric Bin Packing

In this section we give an extensive survey of the literature related to geometric packing and other related problems.

3.1 Geometric Bin Packing

Two-dimensional geometric bin packing (GBP) is substantially different from the 1-D case. Bansal et al. [BCKS06] showed that 2-D bin packing in general does not admit an APTAS unless P = NP.

On the positive side, there has also been a long sequence of works giving improved approximation algorithms. We refer readers to [LMM02] for a review of several greedy heuristics such as Next Fit Decreasing, First Fit Decreasing, Best Fit Decreasing, Finite Best Strip, Floor-Ceiling algorithm, Finite First Fit, Knapsack Packing algorithm, Finite Bottom-Left, Alternate Directions etc. For the case when we do not allow rotation, until the mid 90's the best known bound was a 2.125 approximation [CGJ82], which was improved by Kenyon and Rémila [KR00] to a $2+\epsilon$ approximation (this follows as a corollary of their APTAS for 2-D strip packing) for any $\epsilon > 0$. Caprara in his break-through paper [Cap02] gave an algorithm for 2-D bin packing attaining an asymptotic approximation ratio of $T_\infty (\approx 1.69103)$.

For the case when rotations are allowed, Miyazawa and Wakabayashi [MW04] gave an algorithm with an asymptotic performance guarantee of 2.64. Epstein and Stee [EvS05b] improved it to 2.25 by giving a packing where, in almost all bins, an area of 4/9 is occupied. Finally an asymptotic approximation guarantee arbitrarily close to 2 followed from the result of [JvS05]. This was later improved by Bansal et al. [BCS09] to $(\ln(T_\infty)+1) \approx 1.52$, for both the cases with and without rotation, introducing a novel randomized rounding based framework: *Round and Approx (R & A)* framework. Jansen and Prädel [JP13] improved this guarantee further to give a 1.5-approximation algorithm. Their algorithm is based on exploiting several non-trivial structural properties of how items can be packed in a bin.

Very recently, Bansal and Khan [BK14] gave a polynomial time algorithm with an asymptotic approximation ratio of $\ln(1.5) + 1 \approx 1.405$ for 2-D GBP. This is the best algorithm known so far, and holds both for the case with and without rotations. The main idea behind this result is to show that the *Round and Approx* (R&A) framework introduced by Bansal, Caprara and Sviridenko [BCS09] (See section 2.4.4) can be applied to the $(1.5+\epsilon)$-approximation result of Jansen and Prädel [JP13]. They give a more general argument to apply the R&A framework directly to a wide class of algorithms. In particular, they show that *any* algorithm based on rounding the (large) items into $O(1)$ types, is subset-oblivious. The main observation is that any ρ-approximation based on rounding the item sizes, can be related to another configuration LP (on rounded item sizes) whose solution is no worse than ρ times the optimum solution. They also give some results to show the limitations of rounding based algorithms in obtaining better approximation ratios. There are typically two types of rounding: either the size of an item in some coordinate (such as width or height) is rounded up in an instance-oblivious way (e.g., Harmonic rounding [LL85, Cap02], or Geometric rounding [KK82]), or it is rounded up in an input sensitive way (e.g. linear grouping [dlVL81]). They show that any rounding based algorithm that rounds at least one side of each large item to some number in a constant-size collection values chosen independent of problem instance (let us call such rounding *input-agnostic*), cannot have an approximation ratio better than 3/2. For arbitrary constant rounding based algorithms they show a hardness of 4/3.

We remark that there is still a huge gap between these upper bounds and known lower bounds. In particular, the best known explicit lower bound on the asymptotic approximation for 2-D BP is currently $1 + 1/3792$ and $1 + 1/2196$, for the versions with and without rotations, respectively [CC06]. The best asymptotic worst-case ratio that is achievable in polynomial time for d-dimensional GBP for $d > 2$ is T_∞^{d-1} [Cap02], and in fact it can be achieved by an online algorithm using bounded space. There are no known explicit better lower bounds for higher dimensions.

In *non-asymptotic setting*, without rotations there are a 3-approximation algorithms by Zhang [Zha05] and also by Harren and van Stee [HvS12]. Harren and van Stee [HvS08] gave a non-asymptotic 2-approximation with rotations. Independently this approximation guarantee is also achieved for the version without rotations by Harren and van Stee [HvS09] and Jansen et al. [JPS09]. These 2-approximation results match the non-asymptotic lower bound for this problem, unless $\mathsf{P} = \mathsf{NP}$.

3.2 Square Packing

Leung et al. [LTW$^+$90] have shown that even the special case of packing squares into square is still NP-hard. Kohayakawa et al. [KMRW01] gave a $(2 - (2/3)^d + \epsilon)$ approximation for packing d-dimensional cubes into unit cubes. Later Bansal et al. [BCKS06] have given an APTAS for the problem of packing d-dimensional cubes into d-dimensional unit cubes.

3.3 Online Packing

Coppersmith and Raghavan [CR89] first studied online 2-D GBP and gave algorithms with asymptotic performance ratio of 3.25 and 6.35 for $d = 2$ and 3 respectively. Csirik and van Vliet [CVV93] gave an algorithm with performance ratio T_∞^d (This gives 2.859 for 2-D) for arbitrary dimension d. Epstein and van Stee [EvS05b] achieved the same ratio of T_∞^d only using bounded space and showed it to be the optimal among all bounded space algorithms. In 2002, Seiden and van Stee [SvS02] proposed an elegant algorithm called $H \otimes C$, comprised of the Harmonic algorithm H and the Improved Harmonic algorithm C, for the 2-D online bin packing problem and proved that the algorithm has an asymptotic competitive ratio of at most 2.66013. Since the best known online algorithm for one-dimensional bin packing is the Super Harmonic algorithm [Sei02], a natural question was whether a better upper bound could be achieved by using the

Super Harmonic algorithm instead of the Improved Harmonic algorithm? Han et al. [HCT+11] gave a positive answer for this question and a new upper bound of 2.5545 is obtained for the two-dimensional online bin packing. The main idea is to develop new weighting functions for the Super Harmonic algorithm and propose new techniques to bound the total weight in a rectangular bin. The best known lower bound is 1.907 by Blitz, van Vliet and Woeginger [BvVW96]. We refer the readers to [vS15] for a survey of online algorithms for geometric bin packing in multiple dimensions.

When we allow rotation, Epstein [Eps03b] gave an algorithm with asymptotic performance ratio of 2.45. Later Epstein and van Stee [EVS05c] gave an algorithm with asymptotic performance ratio of 2.25.

For the special case where items are squares, there is also a large number of results. Coppersmith and Raghavan [CR89] showed their algorithm gives asymptotic performance ratio of 2.6875 in this case. They also gave a lower bound of 4/3. Seiden and van Stee [SvS02] gave an algorithm with asymptotic performance ratio of 2.24437. Epstein and van Stee [EvS05a] have shown an upper bound of 2.2697 and a lower bound of 1.6406 for online square packing, and an upper bound of 2.9421 and a lower bound of 1.6680 for online cube packing. The upper bound for squares can be further reduced to 2.24437 using a computer-aided proof. Later Han et al. [HYZ10] get an upper bound of 2.1187 for square packing and 2.6161 for cube packing. For bounded space online algorithms, Epstein and van Stee [EVS07] showed lower and upper bounds for optimal online bounded space hypercube packing till dimensions 7. In particular, for 2-D it lies in $(2.3634, 2.3692)$ and for 3-D it lies in $(2.956, 3.0672)$.

Table 2: Present state of the art for geometric bin packing

Problem	Dim.	Subcase	Best algorithm	Best lower bound
Geometric Bin Packing	2	OFF-REC-WR	asymp[1]: 1.405 [BK14]	$1 + 1/3792$ [CC06]
			abs[2]: 2 [HvS08]	2 (folklore)
		OFF-REC-NR	asymp: 1.405 [BK14]	$1 + 1/2196$ [CC06]
			abs: 2 [HvS09]	2 (folklore)
	d	OFF-REC-NR	asymp: T_∞^{d-1} for $d > 2$ [Cap02]	$1 + 1/2196$ [CC06]
		OFF-CUB	asymp: PTAS[BCKS06]	NP-hard
			abs: 2 [BCKS06]	2 [FMW98]
	2	ON-REC-NR	asymp: 2.5545 [HCT+11]	1.907 [BvVW96]
		ON-REC-WR	asymp: 2.25 [EVS05c]	1.6406 [EvS05a]
		ON-CUB	asymp: 2.1187[HYZ10]	1.6406 [EvS05a]
	3	ON-REC-NR	asymp: 4.3198 [HCT+11]	1.907 [BvVW96]
		ON-CUB	asymp: 2.6161[HYZ10]	1.6680 [EvS05a]

Table 2 summarizes present best approximation/inapproximability results for geometric bin packing. Here OFF denotes offline, ON denotes online, REC denotes rectangles, CUB denotes cubes, WR denotes with rotation and NR denotes without rotation.

3.4 Heuristics

Lodi et al. have reviewed several exact algorithms based on enumerative approach or branch-and-bound in [LMM02]. They have also studied several integer programming models for 2-D

[1]Here asymp. means asymptotic approximation guarantee

[2]Here abs. means absolute approximation guarantee

GBP and other related problems. Pisinger and Sigurd [PS07] gave an algorithm based on Dantzig-Wolfe decomposition. Here the master problem is a set covering problem, solved by delayed column generation. The subproblem deals with the packing of rectangles into a single bin and is solved as a constraint-satisfaction problem (CSP). Martello and Vigo [MV98] had considered exact solutions for 2-D GBP for instance sizes up to 120.

Jylanki [Jyl10] reviewed several greedy heuristics in detail and did an empirical study. Primarily he considered five broad classes of algorithms: shelf algorithms, fit-based guillotine algorithms, split-based guillotine algorithms, maximal rectangle algorithms and skyline algorithms. Lodi [Lod99] also has surveyed several one-phase, two-phase and non-shelf algorithms.

In the past two decades, many local search and meta-heuristic algorithms for rectangle packing problems have been proposed. We refer interested readers to Aarts and Lenstra [AL03] and Glover and Laguna [GL99] for a detailed survey. Dowsland [Dow93] proposed a meta-heuristic approach to strip packing using simulated annealing. During the search, the objective is to minimize pair-wise overlapping area of rectangles and the neighborhood contains all solutions representing vertical or horizontal items shifting. Jakobs [Jak96] presented a genetic algorithm, based on the representation of packing pattern by an order given by some permutation, and packing positions are determined by a Bottom-Left strategy. There are many similar algorithms based on a *permutation coding scheme*. These algorithms consist of two phases: (1) first, find a good permutation using some meta-heuristic, (2) then the decoding algorithm puts items following the permutation order. Several interesting coding schemes have been proposed such as n-leaf binary tree [JvS79], sequence pair [MFNK96], bounded sliceline grid (BCG) [NFMK98], O-tree [GTCY01], B^*-tree [CCWW00], quarter-state sequence [SKM03] etc. Pasha [Pas03] has studied geometric bin packing algorithms for arbitrary shapes and presented several genetic algorithms and simulated annealing based approach. Kroger [Krö95] introduced a sequential and a parallel genetic algorithm based on guillotine cuts.

Lodi et al. [LMV02] introduced a Tabu Search framework exploiting a new constructive heuristic for the evaluation of the neighborhood for 3-D GBP. Kell and Hoeve [KvH13] investigated the application of multivalued decision diagrams (MDDs) to multidimensional bin packing problems. Faroe et al. [FPZ03] have given a guided local search based algorithm for 3-D GBP. For other heuristics for cutting and packing related problems, see the application-oriented research bibliography by Sweeney and Paternoster [SP92]. For further details on meta-heuristics for rectangle packing we refer the readers to [IYN07, HT01, Bor06].

For practical problems such as pallet packing in warehouses, several other factors are needed to be considered such as the stability of packing (under gravity), elasticity, interlocking etc. Several heuristics are considered in [SBS+10, DMO+12] for the stable pallet packing problem. In VLSI design, simulated annealing algorithms are used in practice to solve 3-D Bin Packing problem. Though these algorithms do not have a good worst-case guarantee, they still sometimes work well in practice. There are multiple ways e.g., sequence triple, transitive closure, 3D-CBL [MHDC05] to map each solution of bin packing to a list of 0-1 integers to apply simulated annealing along with different ways to move in the solution space.

3.5 Resource Augmentation

Due to pathological worst-case examples, bin packing has been well-studied under *resource augmentation*, i.e., the side length of the bin is augmented to $(1+\epsilon)$ instead of one. This is also known as *bin stretching*. Though 2-D GBP does not admit an APTAS, Bansal et al. [BCKS06] gave a polynomial time algorithm to pack rectangles into at most m number of bins of size $(1+\epsilon) \times (1+\epsilon)$ where m is the optimal number of unit bins needed to pack all items. Later Bansal and Sviridenko [BS07] showed that this is possible even when we relax the size of the bin in only one dimension.

3.6 Strip Packing

A closely related problem is the *Strip Packing* problem. It is another generalization of the one dimensional bin packing problem and closely tied with the geometric bin packing problem. As we had stated earlier, the best approximation algorithm for 2-D GBP used to be a factor $2 + \epsilon$ and was a corollary from the APTAS for 2-D strip packing due to Kenyon and Rémila [KR00]. The 2-D variant, where we are given a strip with width one and unlimited height and the goal is to pack 2-D rectangular items into the strip so as to minimize the height of the packing, is also known as the *Cutting Stock Problem*. In three dimensions, we are given 3-D rectangular items each of whose dimensions is at most one and they need to be packed into a single 3-D box of unit depth, unit width and unlimited height so as to minimize the height of the packing.

First let us discuss offline algorithms for 2-D strip packing. Baker et al. [BJR80] introduced the problem in 1980 and gave an algorithm with absolute approximation ratio of 3. Later Coffman et al. [JGJT80] introduced Next-Fit Decreasing Height (NFDH), First-Fit Decreasing Height (FFDH) [JGJT80] for 2-D strip packing without rotations, achieving asymptotic approxiamtion ratio as 2 and 1.7 respectively. The upper bound of 2 for NFDH remains valid even for the case with rotations, since the proofs use only area arguments. Epstein and Stee [EvS05b] gave a 3/2 approximation algorithm for the problem with rotation. Finally an APTAS was given for 2-D strip packing without rotations [KR00] and with rotations in [JvS05] using a nice interplay of techniques like fractional strip packing, linear grouping and a variant of NFDH. For the absolute approximation, Harren et al. [HJPvS14] have given a $(5/3 + \epsilon)$-approximation whereas the lower bound is 3/2 which follows from one dimensional bin packing. Very recently, Nadiradze et al. [NW16] have given a $(1.4 + \epsilon)$-absolute approximation algorithm with pseudo-polynomial running time.

Now we discuss online algorithms for 2-D strip packing. Baker and Schwartz [BS83] showed that First-Fit Shelf has asymptotic performance ratio 1.7. Csirik and Woeginger [CW97] improved it to $T_\infty \approx 1.691$ using the *Harmonic* algorithm as a subroutine. They also mention a lower bound of 1.5401. For the absolute performance ratio, Brown et al. [BBK82] have given a lower bound of 2.

3-D strip packing is a common generalization of both the 2-D bin packing problem (when each item has height exactly one) and the 2-D strip packing problem (when each item has width exactly one). Li and Cheng [LC90] were among the first people who considered the problem. They showed 3-D versions of FFDH and NFDH have unbounded worst-case ratio. They gave a 3.25 approximation algorithm, and later gave an online algorithm with upper bound of $T_\infty^2 \approx 2.89$ [LC92] using the Harmonic algorithm as a subroutine. Bansal et al. [BHI+07] gave a 1.69 approximation for the offline case. Recently Jansen and Prädel [JP14] further improved it to 1.5. Both these two algorithms extend techniques from 2-D bin packing.

3.7 Shelf and Guillotine Packing

For $d = 2$, many special structures of packings have been considered in the literature, because they are both easy to deal with and important in practical applications. Among these, very famous are the two-stage packing structures, leading to two-dimensional shelf bin packing (2SBP) and two-dimensional shelf strip packing (2SSP). Two-stage packing problems were originally introduced by Gilmore and Gomory [20] and, thinking in terms of cutting instead of packing, requires that each item be obtained from the associated bin by at most two stages of cutting.

In two-stage packing, in the first stage, the bins are horizontally cut into shelves. The second stage produces slices, which contain a single item by cutting the shelves vertically. Finally, an additional stage (called trimming) is allowed in order to separate an item from a waste area. See Figure 4 for an example of two-stage packing. Two-stage packing is equivalent to packing the items into the bins in shelves, where a *shelf* is a row of items having their bases on a line that is either the base of the bin or the line drawn at the top of the highest item packed in

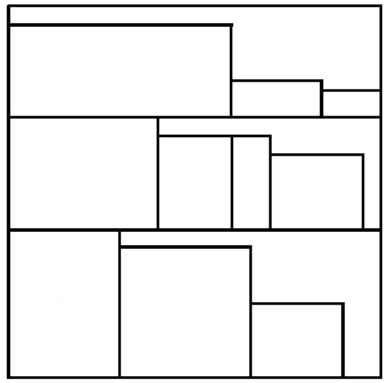

Figure 4: Example of two-stage packing

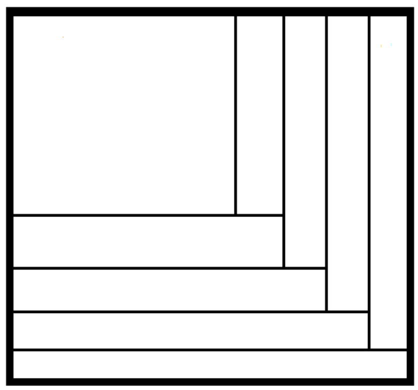

Figure 5: Example of guillotine packing

the shelf below. Formally, a shelf is a set S of items such that total width $\sum_{j \in S} w_j \leq 1$; its height $h(S)$ is given by $max_{j \in S} h_j$. Many classical heuristics for 2-D strip packing ([JGJT80], [BS83], [CW97]) and 2-D GBP ([CGJ82]), including NFDH and FFDH, construct solutions that are in fact feasible for the two-stage versions. Moreover, Caprara et al. [CLM05] presented an APTAS, each for 2SBP and 2SSP. Given this situation, it is natural to ask for the asymptotic worst-case ratio of general packing versus two-stage packing. Csirik and Woeginger [CW97] showed ratio of 2SSP versus 2-D strip packing is equal to T_∞. Caprara [Cap02] showed the ratio of 2SBP versus 2-D GBP is also equal to T_∞. Both their algorithms are online and based on *Harmonic Decreasing Height* (HDH) heuristic. Now consider the optimal 2SBP solution in which the shelves are horizontal as well as the optimal 2SBP solution in which they are vertical. (Recall that near-optimal 2SBP solutions can be found in polynomial time [CLM05].) There is no evidence that the asymptotic worst-case ratio between the best of these two solutions and the optimal 2-D GBP can be as bad as T_∞, and in fact Caprara conjectured that this ratio is 3/2. On the other hand, he also mentions that there are examples where we cannot do better than T_∞, if both solutions are formed by the HDH algorithm in [Cap02].

Seiden and Woeginger [SW05] observed that the APTAS of Kenyon and Rémila [KR00] can easily be adapted to produce a near-optimal packing in three stages for 2-D strip packing, showing that the asymptotic worst-case ratio of 2-D strip packing versus its k-stage version is 1

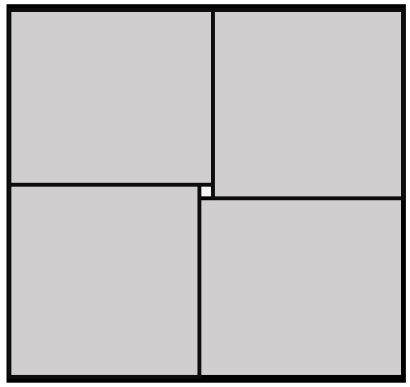

Figure 6: Example of non-guillotine packing

for any $k > 2$, and leading to an APTAS for the latter.

Bansal et al. [BLS05] provided an APTAS for the *guillotine case*, i.e., the case in which the items have to be packed in alternate horizontal and vertical stages but there is no limit on the number of stages that can be used. In the guillotine case, there is a sequence of edge-to-edge cuts parallel to one of the edges of the bin. See Figure 5 for an example of guillotine packing and Figure 6 for an example that is not a guillotine packing. Recently Abed et al. [ACC+15] studied other related packing problems under guillotine cuts. They also made a conjecture that, for any set of n non-overlapping axis-parallel rectangles, there is a guillotine cutting sequence separating $\Omega(n)$ of them. A proof of this conjecture will imply a $O(1)$-approximation for Maximum Independent Set Rectangles, a related NP-hard problem.

3.8 Geometric Knapsack

For 2-D Geometric Knapsack (GK), a result of Steinberg [Ste97] for strip packing translates into a $(3 + \epsilon)$ approximation [CM04]. Present best known approximation algorithm is due to Jansen and Zhang [JZ07] and has an approximation guarantee of $(2+\epsilon)$. On the other hand, no explicit inapproximability results are known. PTAS are known for special cases when resource augmentation is allowed in one dimension [JS07], all items are square [JS08] or all items are small [FGJ05]. Bansal et al. [BCJ+09] gave a PTAS for the special case when the range of the profit-to-area ratio of the rectangles is bounded by a constant. Recently Adamaszek and Wiese [AW15] gave a quasi-PTAS for the problem. This implies that the problem is not APX-hard (thus we can still hope for a PTAS) unless NP $\subseteq$ QP. Very recently, Abed et al. [ACC+15] obtained another quasi-PTAS for the version with guillotine cut.

For 3-D, Diedrich et al. [DHJ+08] have given $7 + \epsilon$ and $5 + \epsilon$ approximation, for the cases without and with rotations, respectively.

Table 3 summarizes present best results for strip packing and geometric knapsack. As previously, OFF denotes offline, ON denotes online, REC denotes rectangles, CUB denotes cubes, WR denotes with rotation and NR denotes without rotation.

[3]See [vS15] for the modified algorithm

Table 3: Present state of the art for strip packing and geometric knapsack

Problem	Dim.	Subcase	Best algorithm	Best lower bound
Strip Packing	2	OFF-REC-WR	asymp: PTAS[JvS05]	NP-hard
			abs: $\frac{5}{3}+\epsilon$ [HJPvS14]	3/2
		OFF-REC-NR	asymp: PTAS[KR00]	NP-hard
	3	OFF-REC-NR	asymp: 1.5 [JP14]	$1+1/2196$ [CC06]
		OFF-CUB	asymp: PTAS[BHI+07]	NP-hard
	2	ON-REC-NR	asymp: T_∞ [CW97]	1.5401 [CW97]
	3	ON-REC-NR	asymp: 2.5545 [HCT+11] [3]	1.907 [BvVW96]
	$d>3$	ON-REC-NR	asymp: T_∞^d [CVV93]	$\to 3$ (for $d\to\infty$) [vS15]
Geometric Knapsack	2	OFF-REC-NR	$2+\epsilon$ [JZ07]	NP-hard
		OFF-CUB	PTAS[JS08]	NP-hard
	3	OFF-REC-NR	$7+\epsilon$ [DHJ+08]	NP-hard
		OFF-REC-WR	$5+\epsilon$ [DHJ+08]	NP-hard

4 Vector Bin Packing

In this section we survey the previous work on vector packing and its variants.

4.1 Offline Vector Packing

The first paper to obtain an APTAS for 1-D bin packing by Fernandez de la Vega and Lueker [dlVL81], implies a $(d+\epsilon)$ approximation for vector packing problem. Woeginger [Woe97] showed that there exists no APTAS even for $d=2$ unless P = NP. However some restricted class of vectors may still admit an APTAS. For example, consider the usual partial order on d dimensional vectors, where $(x_1, x_2, \dots, x_d) \prec (y_1, y_2, \dots, y_d)$ if and only if $x_i \le y_i$ for all $i \in [d]$. In Woeginger's gadget for the lower bound, the items are pairwise incompatible. The opposite extreme case, when there is a total order on all items, is easy to approximate. In fact, a slight modification of de la Vega and Lueker [dlVL81] algorithm yields an APTAS for subproblems of d-dimensional VBP with constant Dilworth number. After nearly twenty years, offline results for the general case were improved by Chekuri and Khanna [CK04]. They gave an algorithm with asymptotic approximation ratio of $(1+\epsilon d+H_{1/\epsilon})$ where $H_k = 1+1/2+\cdots+1/k$, is the k'th Harmonic number. Considering $\epsilon = 1/d$, they show that for fixed d, vector bin packing can be approximated to within $O(\ln d)$ in polynomial time. Bansal, Caprara and Sviridenko [BCS09] then introduced the *Round and Approx* framework and the notion of subset oblivious algorithm and improved it further to $(1+\ln d)$. Both these algorithms run in time that is exponential in d (or worse). Yao [Yao80] showed that no algorithm running in time $o(n \log n)$ can give better than a d-approximation.

For arbitrary d, Chekuri-Khanna [CK04] showed vector bin packing is hard to approximate to within a $d^{1/2-\epsilon}$ factor for all fixed $\epsilon > 0$ using a reduction from graph coloring problem. This can be improved to $d^{1-\epsilon}$ by using the following simple reduction. Let G be a graph on n vertices. In the d-dimensional VBP instance, there will be $d=n$ dimensions and n items, one for each vertex. For each vertex i, we create an item i that has size 1 in coordinate i and size $1/n$ in coordinate j for each neighbor j of i, and size 0 in every other coordinate. It is easily verified that a set of items S can be packed into a bin if and only if S is an independent set in

G. Thus we mainly focus on the case when d is a fixed constant and not part of the input.

The two dimensional case has received special attention. Kellerer and Kotov [KK03] designed an algorithm for 2-D vector packing with worst case absolute approximation ratio as 2. On the other hand there is a hardness of 3/2 for absolute approximation ratio that comes from the hardness of 1-D bin packing.

Very recently, Bansal, Elias and Khan [BEK16] have given improved approximation for multidimensional vector packing. They give a polynomial time algorithm with an asymptotic approximation ratio of $(1 + \ln(1.5) + \epsilon) \approx (1.405 + \epsilon)$ for 2-D vector packing and a $\ln d + 0.807 + o_d(1) + \epsilon$-approximation for d-dimensional vector packing. This overcomes a natural barrier of $(1 + \ln d)$ of R&A framework due to the fact that one cannot obtain better than d-approximation using rounding based algorithms. They circumvent this problem based on two ideas.

First, they show a structural property of vector packing that any optimal packing of m bins can be transformed into nearly $\lceil \frac{3m}{2} \rceil$ bins of two types:
1. Either a bin contains at most two big items, or
2. The bin has slack in one dimension (i.e., the sum of all vectors in the bin is at most $1 - \delta$ for some constant δ). They then search (approximately) over the space of such "well-structured" 1.5-approximate solutions. However, as this structured solution (necessarily) uses unrounded item sizes, it is unclear how to search over the space of such solutions efficiently. So a key idea is to define this structure carefully based on matchings, and use an elegant recent algorithm for the multiobjective-multibudget matching problem by Chekuri, Vondrák, and Zenklusen [CVZ11].

The second step is to apply the subset oblivious framework to the above algorithm. There are two problems. First, the algorithm is not rounding-based. Second, even proving subset obliviousness for rounding based algorithms for vector packing is more involved than for geometric bin-packing. To get around these issues, they use additional technical observations about the structure of d-dimensional VBP.

Another consequence of the these techniques is the following tight (absolute) approximation guarantee. they show that for any small constant $\epsilon > 0$, there is a polynomial time algorithm with an almost tight absolute approximation ratio of $(1.5 + \epsilon)$ for 2-D vector packing.

4.2 Online Vector Packing

A generalization of the First Fit algorithm by Garey et al. [GGJ76] gives $d + \frac{7}{10}$ competitive ratio for the online version. Galamobos et al. [GKW93] showed a lower bound on the performance ratio of online algorithms that tends to 2 as d grows. The gap persisted for a long time, and in fact it was conjectured in [Eps03a] that the lower bound is super constant, but sublinear.

Recently Azar et al. [ACKS13] settled the status by giving $\Omega(d^{1-\epsilon})$ information theoretic lower bound using stochastic packing integer programs and online graph coloring. In fact their result holds for arbitrary bin size $B \in \mathbb{Z}^+$ if the bin is allowed to grow. In particular, they show that for any integer $B \geq 1$, any deterministic online algorithm for VBP has a competitive ratio of $\Omega(d^{\frac{1}{B} - \epsilon})$. For $\{0, 1\}$-VBP the lower bound is $\Omega(d^{\frac{1}{B+1} - \epsilon})$. They also provided an improved upper bound for $B \geq 2$ with a polynomial time algorithm for the online VBP with competitive ratio: $O(d^{1/(B-1)} \log d^{B/(B+1)})$, for $[0, 1]^d$ vectors and $O(d^{1/B} \log d^{(B+1)/B})$, for $\{0, 1\}^d$ vectors. Very recently, Azar et al. [ACFR16] studied the online vector packing for small vectors (relative to the size of a bin). For this special case, they give a constant competitive ratio of e for arbitrary d. For 2-D, they present a First Fit variant with a competitive ratio ≈ 1.48 and another essentially tight algorithm (not via a First Fit variant) with a competitive ratio arbitrarily close to 4/3.

4.3 Vector Scheduling

For d-dimensional vector scheduling, the first major result was obtained by Chekuri and Khanna [CK04]. They obtained a PTAS when d is a fixed constant, generalizing the classical result of Hochbaum and Shmoys [HS87] for multiprocessor scheduling. For arbitrary d, they obtained

$O(\ln^2 d)$-approximation using approximation algorithms for packing integer programs (PIPs) as a subroutine. They also showed that, when m is the number of bins in the optimal solution, a simple random assignment gives $O(\ln dm/\ln\ln dm)$-approximation algorithm which works well when m is small. Furthermore, they showed that it is hard to approximate within any constant factor when d is arbitrary. This $\omega(1)$ lower bound is still the present best lower bound for the offline case.

In the online setting, Meyerson et al. [MRT13] gave deterministic online algorithms with $O(\log d)$ competitive ratio. Im et al. [IKKP14] recently gave an algorithm with $O(\log d/\log\log d)$-competitive ratio. They also show tight information theoretic lower bound of $\Omega(\log d/\log\log d)$. Surprisingly this is also the present best offline algorithm!

4.4 Vector Bin Covering

For d-dimensional vector bin covering problem Alon et al. [AAC⁺98] gave an online algorithm with competitive ratio $\frac{1}{2d}$, for $d \geq 2$, and they showed an information theoretic lower bound of $\frac{2}{2d+1}$. For the offline version they give an algorithm with an approximation guarantee of $\Theta(\frac{1}{\log d})$.

Table 4: Present state of the art for vector packing and related variants

Problem	Subcase	Best algorithm	Best lower bound
Vector Bin Packing	Offline (constant d)	$\ln d + 0.807 + o_d(1) + \epsilon$ (asymp.[4]) [BEK16]	APX-hard [Woe97]
	$d = 2$	$1.405 + \epsilon$ (asymp.) [BEK16]	APX-hard [Woe97]
		$3/2 + \epsilon$ (abs.[5]) [BEK16]	$3/2$ [6]
	Offline (arbitrary d)	$1 + \epsilon d + O(\ln \frac{1}{\epsilon})$ [CK04]	$d^{1-\epsilon}$ [7]
	Online	$d + \frac{7}{10}$ [GGJ76]	$\Omega(d^{1-\epsilon})$ [ACKS13]
Vector Scheduling	Offline (constant d)	PTAS[CK04]	NP-hard
	Offline (arbitrary d)	$O(\frac{\log d}{\log\log d})$ [IKKP14]	$\omega(1)$ [CK04]
	Online	$O(\frac{\log d}{\log\log d})$ [IKKP14]	$\Omega(\frac{\log d}{\log\log d})$ [IKKP14]

4.5 Heuristics

Heuristics for 2-D VBP were studied in detail by Spieksma [Spi94], who mentions applications in loading, scheduling, and layout design, considers lower bounding and heuristic procedures using a branch-and-bound scheme. Here, upper bounds are derived from a heuristic, adapted from the first fit decreasing (FFD)-rule for bin-packing. To find better lower bounds, properties of pairs of items are investigated. Han et al. [HDC94] present heuristic and exact algorithms for a variant of 2-D VBP, where the bins are not identical. Caprara and Toth [CT01] also studied 2-D VBP. They analyze several lower bounds for the 2-D VBP. In particular, they determine an upper bound on the worst-case performance of a class of lower bounding procedures derived from the classical 1-D BP. They also prove that the lower bound associated with the huge LP relaxation dominates all the other lower bounds. They then introduce heuristic and exact algorithms, and

[4] Here asymp. means asymptotic approximation guarantee

[5] Here abs. means absolute approximation guarantee

[6] Follows from the fact that even 1-D bin packing cannot be approximated better than 3/2

[7] See the reduction in Section 4.1

report extensive computational results on several instance classes, showing that in some cases the combinatorial approach allows for a fast solution of the problem, while in other cases one has to resort to a large formulation for finding optimal solutions. Chang et al. [CHP05] had proposed a greedy heuristic named *hedging*. Otoo et al. [OPR11] studied the 2-D VBP, where each item has 2 distinct weights and each bin has 2 corresponding capacities, and have given linear-time greedy heuristics. An interesting application of the 2-D VBP problem is studied by Vercruyssen and Muller [VM87]. The application arises in a factory where coils of steel plates (items), each having a certain physical weight and height, have to be distributed over identical furnaces (bins) with a limited capacity for height and weight. Another application of the problem is described by Sarin and Wilhelm [SW84], in the context of layout design. Here, a number of machines (items) have to be assigned to a number of robots (bins), with each robot having a limited capacity for space, as well as a limited capacity for serving a machine. Many of these heuristics are tailor-made for 2-D.

For the general case, Stillwell et al. [SSVC10] studied variants of FFD concluding that the algorithm *FFDAvgSum* is best in practice. They also show that genetic algorithms do not perform well. Panigrahy et al. [PTUW11] systematically studied variants of the First Fit Decreasing (FFD) algorithm. Inspired by bad instances for FFD-type algorithms, they propose new geometric heuristics that run nearly as fast as FFD for reasonable values of n and d.

5 Open Problems

In this section we conclude by listing ten major open problems related to multidimensional bin packing.

Problem 1. Tight approximability of bin packing.
The present best algorithm for 1-D BP by Hoberg and Rothvoß [HR15], uses $\mathsf{Opt} + O(\log \mathsf{Opt})$ bins. Proving one could compute a packing with only a constant number of extra bins will be a remarkable progress and is mentioned as one of the ten most important problems in approximation algorithms [WS11]. Consider the seemingly simple 3-Partition case in which all n items have sizes $s_i \in (1/4, 1/2)$. Recent progress by [NNN12] suggests that either $O(\log n)$ bound is the best possible for 3-Partition or some fundamentally new ideas are needed to make progress.

Problem 2. Integrality gap of Gilmore-Gomory LP.
It has been conjectured in [ST97] that the Gilmore-Gomory LP for 1-D BP has *Modified Integer Roundup Property*, i.e., $\mathsf{Opt} \leq \lceil \mathsf{Opt}_f \rceil + 1$. The conjecture has been proved true for the case when the instance contains at most 7 different item sizes [SS09]. Settling the status for the general case is an important open problem in optimization.

Problem 3. Tight asymptotic competitive ratio for 1-D online BP.
The present best algorithm for online bin packing is by Heydrich and van Stee [HvS15] who presented an online algorithm with asymptotic performance ratio of at most 1.5817 using a new type of *interval classification*. They also gave a lower bound of 1.5766 for any interval classification algorithm. In general the best known lower bound for asymptotic competitive ratio is 1.54014 [vV92]. Giving a stronger lower bound using some other construction is an important question in online algorithms.

Problem 4. Improved approximability for geometric bin packing.
There is a huge gap between the best approximation guarantee and hardness of geometric bin packing. There are no explicit inapproximability bounds known for multidimensional bin packing as function of d, apart from the APX-hardness in 2-D. Thus there is a huge gap between the best algorithm (1.69^{d-1}, i.e., exponential in d) and the hardness. Improved inapproximability,

as a function of d, will be an interesting hardness result.

Problem 5. Improved approximability for vector bin packing.
Similarly, there are no explicit inapproximability bounds known for vector bin packing as function of d, apart from the APX-hardness in 2-D. Thus there is a gap between the best algorithm ($O(\ln d)$ for vector packing for $d > 2$) and the hardness. Improved inapproximability, as a function of d, will be an interesting hardness result even in this case.

Problem 6. Improved approximability for geometric knapsack.
Finding a PTAS for 2-D geometric knapsack is one of the major problems related to bin packing.

Problem 7. Tight ratio between optimal Guillotine packing and optimal bin packing. Improving the present guarantee for 2-D GBP will require an algorithm that is not *input-agnostic*. In particular, this implies that it should have the property that it can round two identical items (i.e., with identical height and width) differently. One such candidate is the guillotine packing approach [BLS05]. It has been conjectured that this approach can give an approximation ratio of 4/3 for 2-D GBP. At present the best known upper bound on this gap is $T_\infty \approx 1.69$ [CLM05]. Guillotine cutting also has connections with other geometric packing problems such as GEOMETRIC KNAPSACK and MAXIMUM INDEPENDENT SET RECTANGLES [ACC+15].

Problem 8. Tight ratio between optimal two-stage packing and optimal bin packing.
Caprara conjectured [Cap02] that there is a two-stage packing that gives 3/2 approximation for 2-D bin packing. As there are PTAS for 2-stage packing [CLM05], this will give another 3/2 approximation for 2-D BP and coupled with our R&A method this will give another $(1.405 + \epsilon)$ approximation. Presently the upper bound between best two-stage packing and optimal bin packing is $T_\infty \approx 1.69$. As 2-stage packings are very well-studied, this question is of independent interest and it might give us more insight on the power of Guillotine packing.

Problem 9. Extending R&A framework to d-D GBP and 3-D SP.
One key bottleneck to extend R&A framework to d-D GBP or other related problems, is to find a good approximation algorithm to find the solution of the configuration LP. A $poly(d)$ asymptotic approximation for the LP will give us a $poly(d)$ asymptotic approximation for d-D GBP, a significant improvement over the current best ratio of $2^{O(d)}$ for $d > 2$.

Problem 10. Tight absolute approximation for 2-D SP.
As we had earlier mentioned, there is a gap between the best upper bound of $(5/3+\epsilon)$ [HJPvS14] and lower bound of 3/2. Tightening the gap is an interesting open problem.

Finally, finding faster heuristics that work well in practice, is also a very important problem.

References

[AAC+98] Noga Alon, Yossi Azar, János Csirik, Leah Epstein, Sergey V. Sevastianov, Arjen P. A. Vestjens, and Gerhard J. Woeginger. On-line and off-line approximation algorithms for vector covering problems. *Algorithmica*, 21(1):104–118, 1998. 22

[ACC+15] Fidaa Abed, Parinya Chalermsook, José R. Correa, Andreas Karrenbauer, Pablo Pérez-Lantero, José A. Soto, and Andreas Wiese. On guillotine cutting sequences. In *APPROX*, pages 1–19, 2015. 19, 24

[ACFR16] Yossi Azar, Ilan Cohen, Amos Fiat, and Alan Roytman. Packing small vectors. *To appear in SODA*, 2016. 21

[ACKS13] Yossi Azar, Ilan Reuven Cohen, Seny Kamara, and Bruce Shepherd. Tight bounds for online vector bin packing. In *STOC*, pages 961–970, 2013. 21, 22

[AHK12] Sanjeev Arora, Elad Hazan, and Satyen Kale. The multiplicative weights update method: a meta-algorithm and applications. *Theory of Computing*, 8(1):121–164, 2012. 7

[AL03] Emile HL Aarts and Jan Karel Lenstra. *Local search in combinatorial optimization*. Princeton University Press, 2003. 16

[AW15] Anna Adamaszek and Andreas Wiese. A quasi-ptas for the two-dimensional geometric knapsack problem. In *SODA*, pages 1491–1505, 2015. 19

[BBD+15] János Balogh, József Békési, György Dósa, Jirí Sgall, and Rob van Stee. The optimal absolute ratio for online bin packing. In *SODA*, pages 1425–1438, 2015. 7

[BBK82] Donna J Brown, Brenda S Baker, and Howard P Katseff. Lower bounds for on-line two-dimensional packing algorithms. *Acta Informatica*, 18(2):207–225, 1982. 17

[BC81] Brenda S. Baker and Edward G. Coffman, Jr. A tight asymptotic bound for next-fit-decreasing bin-packing. *SIAM Journal on Algebraic Discrete Methods*, 2(2):147–152, 1981. 6

[BCJ+09] Nikhil Bansal, Alberto Caprara, Klaus Jansen, Lars Prädel, and Maxim Sviridenko. A structural lemma in 2-dimensional packing, and its implications on approximability. In *ISAAC*, pages 77–86, 2009. 11, 13, 19

[BCKS06] Nikhil Bansal, José R. Correa, Claire Kenyon, and Maxim Sviridenko. Bin packing in multiple dimensions: Inapproximability results and approximation schemes. *Math. Oper. Res.*, 31(1):31–49, 2006. 10, 13, 15, 16, 17

[BCS09] Nikhil Bansal, Alberto Caprara, and Maxim Sviridenko. A new approximation method for set covering problems, with applications to multidimensional bin packing. *SIAM J. Comput.*, 39(4):1256–1278, 2009. 12, 13, 14, 20

[BEK16] Nikhil Bansal, Marek Elias, and Arindam Khan. Improved approximation for vector packing. *To appear in SODA*, 2016. 13, 21, 22

[BF07] Joan Boyar and Lene M Favrholdt. The relative worst order ratio for online algorithms. *ACM Transactions on Algorithms (TALG)*, 3(2):22, 2007. 6

[BHI+07] Nikhil Bansal, Xin Han, Kazuo Iwama, Maxim Sviridenko, and Guochuan Zhang. Harmonic algorithm for 3-dimensional strip packing problem. In *SODA*, pages 1197–1206, 2007. 17, 20

[BJR80] Brenda S. Baker, Edward G. Coffman Jr., and Ronald L. Rivest. Orthogonal packings in two dimensions. *SIAM J. Comput.*, 9(4):846–855, 1980. 4, 17

[BK14] Nikhil Bansal and Arindam Khan. Improved approximation algorithm for two-dimensional bin packing. In *SODA*, pages 13–25, 2014. 13, 14, 16

[BLN01] Joan Boyar, Kim S Larsen, and Morten N Nielsen. The accommodating function: A generalization of the competitive ratio. *SIAM Journal on Computing*, 31(1):233–258, 2001. 6

[BLS05] Nikhil Bansal, Andrea Lodi, and Maxim Sviridenko. A tale of two dimensional bin packing. In *FOCS*, pages 657–666, 2005. 11, 19, 24

[Bor06] Andreas Bortfeldt. A genetic algorithm for the two-dimensional strip packing problem with rectangular pieces. *European Journal of Operational Research*, 172(3):814–837, 2006. 16

[BS83] Brenda S. Baker and Jerald S. Schwarz. Shelf algorithms for two-dimensional packing problems. *SIAM J. Comput.*, 12(3):508–525, 1983. 17, 18

[BS07] Nikhil Bansal and Maxim Sviridenko. Two-dimensional bin packing with one-dimensional resource augmentation. *Discrete Optimization*, 4(2):143–153, 2007. 17

[BvVW96] D. Blitz, A. van Vliet, and G.J Woeginger. Lower bounds on the asymptotic worst-case ratio of online bin packing alorithms. *Manuscript*, 1996. 15, 16, 20

[Cap02] Alberto Caprara. Packing 2-dimensional bins in harmony. In *FOCS*, pages 490–499, 2002. 11, 12, 14, 16, 19, 24

[CC06] Miroslav Chlebík and Janka Chlebíková. Inapproximability results for orthogonal rectangle packing problems with rotations. In *CIAC*, pages 199–210, 2006. 14, 16, 20

[CCWW00] Yun-Chih Chang, Yao-Wen Chang, Guang-Ming Wu, and Shu-Wei Wu. B*-trees: a new representation for non-slicing floorplans. In *Proceedings of the 37th Annual Design Automation Conference*, pages 458–463. ACM, 2000. 16

[CGJ82] F. Chung, M. R. Garey, and David S. Johnson. On packing two-dimensional bins. *SIAM J. Algebraic Discrete Methods*, 3:66–76, 1982. 14, 18

[CGMV99] Edward G. Coffman, Gabor Galambos, Silvano Martello, and Daniele Vigo. Bin packing approximation algorithms: Combinatorial analysis. In *Handbook of combinatorial optimization*, pages 151–207. Springer, 1999. 3

[CHP05] Soo Y Chang, Hark-Chin Hwang, and Sanghyuck Park. A two-dimensional vector packing model for the efficient use of coil cassettes. *Computers & operations research*, 32(8):2051–2058, 2005. 23

[CJCG+13] Edward G. Coffman Jr, János Csirik, Gábor Galambos, Silvano Martello, and Daniele Vigo. Bin packing approximation algorithms: Survey and classification. In *Handbook of Combinatorial Optimization*, pages 455–531. Springer, 2013. 3, 6

[CJGJ84] Edward G. Coffman Jr, Michael R. Garey, and David S Johnson. Approximation algorithms for bin-packin : an updated survey. In *Algorithm design for computer system design*, pages 49–106. Springer, 1984. 3

[CJK+06] Janos Csirik, David S Johnson, Claire Kenyon, James B Orlin, Peter W Shor, and Richard R Weber. On the sum-of-squares algorithm for bin packing. *Journal of the ACM (JACM)*, 53(1):1–65, 2006. 8

[CK04] Chandra Chekuri and Sanjeev Khanna. On multidimensional packing problems. *SIAM J. Comput.*, 33(4):837–851, 2004. 20, 22

[CL91] Edward G. Coffman and George S Lueker. *Probabilistic analysis of packing and partitioning algorithms*. Wiley New York, 1991. 3

[CLM05] Alberto Caprara, Andrea Lodi, and Michele Monaci. Fast approximation schemes for two-stage, two-dimensional bin packing. *Math. Oper. Res.*, 30(1):150–172, 2005. 10, 18, 19, 24

[CM04] Alberto Caprara and Michele Monaci. On the two-dimensional knapsack problem. *Oper. Res. Lett.*, 32(1):5–14, 2004. 19

[CMM71] Richard W. Conway, William L. Maxwell, and Louis W. Miller. Theory of scheduling, 1967. *Addison-Wesley*, 1971. 3

[CP91] Pierluigi Crescenzi and Alessandro Panconesi. Completeness in approximation classes. *Information and Computation*, 93(2):241–262, 1991. 5

[CR89] Don Coppersmith and Prabhakar Raghavan. Multidimensional on-line bin packing: algorithms and worst-case analysis. *Operations Research Letters*, 8(1):17–20, 1989. 15

[CT01] Alberto Caprara and Paolo Toth. Lower bounds and algorithms for the 2-dimensional vector packing problem. *Discrete Applied Mathematics*, 111(3):231–262, 2001. 22

[CVV93] János Csirik and André Van Vliet. An on-line algorithm for multidimensional bin packing. *Operations Research Letters*, 13(3):149–158, 1993. 15, 20

[CVZ11] Chandra Chekuri, Jan Vondrák, and Rico Zenklusen. Multi-budgeted matchings and matroid intersection via dependent rounding. In *SODA*, pages 1080–1097, 2011. 21

[CW97] János Csirik and Gerhard J. Woeginger. Shelf algorithms for on-line strip packing. *Inf. Process. Lett.*, 63(4):171–175, 1997. 17, 18, 20

[CW98] János Csirik and Gerhard J Woeginger. *On-line packing and covering problems.* Springer, 1998. 3

[DGP98] Marc Demange, Pascal Grisoni, and Vangelis Th Paschos. Differential approximation algorithms for some combinatorial optimization problems. *Theoretical Computer Science*, 209(1):107–122, 1998. 6

[DHJ+08] Florian Diedrich, Rolf Harren, Klaus Jansen, Ralf Thöle, and Henning Thomas. Approximation algorithms for 3d orthogonal knapsack. *J. Comput. Sci. Technol.*, 23(5):749–762, 2008. 19, 20

[DLHT13] György Dósa, Rongheng Li, Xin Han, and Zsolt Tuza. Tight absolute bound for first fit decreasing bin-packing: Ffd(l) $\leq$ 11/9 OPT(L) + 6/9. *Theoretical Computer Science*, 510:13–61, 2013. 6

[dlVL81] Wenceslas Fernandez de la Vega and George S. Lueker. Bin packing can be solved within 1+epsilon in linear time. *Combinatorica*, 1(4):349–355, 1981. 6, 11, 12, 14, 20

[DMO+12] Girum Demisse, Razvan Mihalyi, Billy Okal, Dev Poudel, Johannes Schauer, and Andreas Nüchter. Mixed palletizing and task completion for virtual warehouses. In *Virtual Manufacturing and Automation Competition (VMAC) Workshop at the Int. Conference of Robotics and Automation (ICRA)*, 2012. 16

[Dow93] Kathryn A Dowsland. Some experiments with simulated annealing techniques for packing problems. *European Journal of Operational Research*, 68(3):389–399, 1993. 16

[DS13] György Dósa and Jiri Sgall. First fit bin packing: A tight analysis. In *STACS 2013*, pages 538–549, 2013. 6

[EC71] Samuel Eilon and Nicos Christofides. The loading problem. *Management Science*, 17(5):259–268, 1971. 3

[EPR13] Friedrich Eisenbrand, Dömötör Pálvölgyi, and Thomas Rothvoß. Bin packing via discrepancy of permutations. *ACM Transactions on Algorithms*, 9(3):24, 2013. 7

[Eps03a] Leah Epstein. On variable sized vector packing. *Acta Cybern.*, 16(1):47–56, 2003. 21

[Eps03b] Leah Epstein. Two dimensional packing: the power of rotation. In *Mathematical Foundations of Computer Science 2003*, pages 398–407. Springer, 2003. 15

[Eps15] Leah Epstein. Harmonic algorithm for bin packing. In *Encyclopedia of Algorithms.* 2015. 12

[EvS05a] Leah Epstein and Rob van Stee. Online square and cube packing. *Acta Inf.*, 41(9):595–606, 2005. 15, 16

[EvS05b] Leah Epstein and Rob van Stee. Optimal online algorithms for multidimensional packing problems. *SIAM Journal on Computing*, 35(2):431–448, 2005. 14, 15, 17

[EVS05c] Leah Epstein and Rob Van Stee. This side up! In *Approximation and Online Algorithms*, pages 48–60. Springer, 2005. 15, 16

[EVS07] Leah Epstein and Rob Van Stee. Bounds for online bounded space hypercube packing. *Discrete optimization*, 4(2):185–197, 2007. 15

[FC84] AM Frieze and MRB Clarke. Approximation algorithms for the m-dimensional 0–1 knapsack problem: Worst-case and probabilistic analyses. *European Journal of Operational Research*, 15(1):100–109, 1984. 11, 13

[FGJ05] Aleksei V. Fishkin, Olga Gerber, and Klaus Jansen. On efficient weighted rectangle packing with large resources. In *ISAAC*, pages 1039–1050, 2005. 19

[FMW98] Carlos E Fereira, Flavio K Miyazawa, and Yoshiko Wakabayashi. Packing of squares into squares. In *Pesquisa Operacional*. Citeseer, 1998. 16

[FPZ03] Oluf Faroe, David Pisinger, and Martin Zachariasen. Guided local search for the three-dimensional bin-packing problem. *INFORMS Journal on Computing*, 15(3):267–283, 2003. 16

[GG61] Paul C. Gilmore and Ralph E. Gomory. A linear programming approach to the cutting-stock problem. *Operations research*, 9(6):849–859, 1961. 3, 6

[GGJ76] M. R. Garey, Ronald L. Graham, and David S. Johnson. Resource constrained scheduling as generalized bin packing. *J. Comb. Theory, Ser. A*, 21(3):257–298, 1976. 21, 22

[GGU72] M. R. Garey, Ronald L. Graham, and Jeffrey D. Ullman. Worst-case analysis of memory allocation algorithms. In *STOC*, pages 143–150, 1972. 3

[GJ79] Michael R. Garey and David S. Johnson. Computers and interactibility: A guide to the theory of np-completeness. *W.H. Freeman & Co., SF, CA*, 1979. 3

[GJ81] Michael R Garey and David S Johnson. Approximation algorithms for bin packing problems: A survey. In *Analysis and design of algorithms in combinatorial optimization*, pages 147–172. Springer, 1981. 3

[GKPV01] Michael D. Grigoriadis, Leonid G. Khachiyan, Lorant Porkolab, and J. Villavicencio. Approximate max-min resource sharing for structured concave optimization. *SIAM Journal on Optimization*, 11(4):1081–1091, 2001. 11

[GKW93] Gábor Galambos, Hans Kellerer, and Gerhard J. Woeginger. A lower bound for on-line vector-packing algorithms. *Acta Cybern.*, 11(1-2):23–34, 1993. 21

[GL99] Fred Glover and Manuel Laguna. *Tabu search*. Springer, 1999. 16

[GLS88] Martin Grötschel, László Lovász, and Alexander Schrijver. Geometric algorithm and combinatorial optimization. *Algorithms and Combinatorics: Study and Research Texts, 2. Springer-Verlag, Berlin*, 1988. 7, 11

[Gon07] Teofilo F Gonzalez. *Handbook of approximation algorithms and metaheuristics*. CRC Press, 2007. 3, 4

[GR12] Varun Gupta and Ana Radovanovic. Online stochastic bin packing. *arXiv preprint arXiv:1211.2687*, 2012. 8

[GR14] Michel X. Goemans and Thomas Rothvoß. Polynomiality for bin packing with a constant number of item types. In *SODA 2014*, pages 830–839, 2014. 7

[GTCY01] Pei-Ning Guo, Toshihiko Takahashi, Chung-Kuan Cheng, and Takeshi Yoshimura. Floorplanning using a tree representation. *Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on*, 20(2):281–289, 2001. 16

[GW95] Gábor Galambos and Gerhard J. Woeginger. On-line bin packing - A restricted survey. *Math. Meth. of OR*, 42(1):25–45, 1995. 3

[HCT+11] Xin Han, Francis Y. L. Chin, Hing-Fung Ting, Guochuan Zhang, and Yong Zhang. A new upper bound 2.5545 on 2d online bin packing. *ACM Transactions on Algorithms*, 7(4):50, 2011. 15, 16, 20

[HDC94] Bernard T Han, George Diehr, and Jack S Cook. Multiple-type, two-dimensional bin packing problems: Applications and algorithms. *Annals of Operations Research*, 50(1):239–261, 1994. 22

[HJPvS14] Rolf Harren, Klaus Jansen, Lars Prädel, and Rob van Stee. A $(5/3 + \epsilon)$-approximation for strip packing. *Comput. Geom.*, 47(2):248–267, 2014. 17, 20, 24

[Hoc96] Dorit S Hochbaum. *Approximation algorithms for NP-hard problems*. PWS Publishing Co., 1996. 3

[HR15] Rebecca Hoberg and Thomas Rothvoß. A logarithmic additive integrality gap for bin packing. *CoRR*, abs/1503.08796, 2015. 6, 7, 23

[HS87] Dorit S Hochbaum and David B Shmoys. Using dual approximation algorithms for scheduling problems theoretical and practical results. *Journal of the ACM (JACM)*, 34(1):144–162, 1987. 22

[HT01] EBCH Hopper and BCH Turton. An empirical investigation of meta-heuristic and heuristic algorithms for a 2d packing problem. *European Journal of Operational Research*, 128(1):34–57, 2001. 16

[HvS08] Rolf Harren and Rob van Stee. Packing rectangles into 2opt bins using rotations. In *SWAT*, pages 306–318, 2008. 14, 16

[HvS09] Rolf Harren and Rob van Stee. Improved absolute approximation ratios for two-dimensional packing problems. In *APPROX-RANDOM*, pages 177–189, 2009. 14, 16

[HvS12] Rolf Harren and Rob van Stee. Absolute approximation ratios for packing rectangles into bins. *J. Scheduling*, 15(1):63–75, 2012. 14

[HvS15] Sandy Heydrich and Rob van Stee. Beating the harmonic lower bound for online bin packing. *arXiv preprint arXiv:1511.00876*, 2015. 7, 23

[HYZ10] Xin Han, Deshi Ye, and Yong Zhou. A note on online hypercube packing. *Central European Journal of Operations Research*, 18(2):221–239, 2010. 15, 16

[IKKP14] Sungjin Im, Nathaniel Kell, Janardhan Kulkarni, and Debmalya Panigrahi. Tight bounds for online vector scheduling. *CoRR*, abs/1411.3887, 2014. 22

[IYN07] Shinji Imahori, Mutsunori Yagiura, and Hiroshi Nagamochi. Practical algorithms for two-dimensional packing. *Handbook of Approximation Algorithms and Metaheuristics. Chapman & Hall/CRC Computer & Information Science Series*, 13, 2007. 16

[Jak96] Stefan Jakobs. On genetic algorithms for the packing of polygons. *European Journal of Operational Research*, 88(1):165–181, 1996. 16

[JDU+74] David S. Johnson, Alan Demers, Jeffrey D. Ullman, Michael R Garey, and Ronald L. Graham. Worst-case performance bounds for simple one-dimensional packing algorithms. *SIAM Journal on Computing*, 3(4):299–325, 1974. 3, 6

[JGJT80] Edward G. Coffman Jr., M. R. Garey, David S. Johnson, and Robert Endre Tarjan. Performance bounds for level-oriented two-dimensional packing algorithms. *SIAM J. Comput.*, 9(4):808–826, 1980. 4, 10, 17, 18

[JKMS13] Klaus Jansen, Stefan Kratsch, Dániel Marx, and Ildikó Schlotter. Bin packing with fixed number of bins revisited. *Journal of Computer and System Sciences*, 79(1):39–49, 2013. 7

[Joh73] David S Johnson. Approximation algorithms for combinatorial problems. In *STOC*, pages 38–49, 1973. 3

[JP13] Klaus Jansen and Lars Prädel. New approximability results for two-dimensional bin packing. In *SODA*, 2013. 11, 14

[JP14] Klaus Jansen and Lars Prädel. A new asymptotic approximation algorithm for 3-dimensional strip packing. In *SOFSEM 2014: Theory and Practice of Computer Science*, pages 327–338. Springer, 2014. 17, 20

[JPS09] Klaus Jansen, Lars Prädel, and Ulrich M. Schwarz. Two for one: Tight approximation of 2d bin packing. In *WADS*, pages 399–410, 2009. 14

[JS07] Klaus Jansen and Roberto Solis-Oba. New approximability results for 2-dimensional packing problems. In *MFCS*, pages 103–114, 2007. 19

[JS08] Klaus Jansen and Roberto Solis-Oba. A polynomial time approximation scheme for the square packing problem. In *IPCO*, pages 184–198, 2008. 19, 20

[JSO03] Klaus Jansen and Roberto Solis-Oba. An asymptotic fully polynomial time approximation scheme for bin covering. *Theoretical Computer Science*, 306(1):543–551, 2003. 9

[JvS79] Klaus Jansen and Rob van Stee. Placement algorithms for arbitrarily shaped blocks. In *DAC*, page 474, 1979. 16

[JvS05] Klaus Jansen and Rob van Stee. On strip packing with rotations. In *STOC*, pages 755–761, 2005. 14, 17, 20

[Jyl10] Jukka Jylänki. A thousand ways to pack the bin-a practical approach to two-dimensional rectangle bin packing. *retrived from http://clb. demon. fi/files/RectangleBinPack. pdf*, 2010. 16

[JZ07] Klaus Jansen and Guochuan Zhang. Maximizing the total profit of rectangles packed into a rectangle. *Algorithmica*, 47(3):323–342, 2007. 19, 20

[Ken96] Claire Kenyon. Best-fit bin-packing with random order. In *SODA*, volume 96, pages 359–364, 1996. 6

[Kha15] Arindam Khan. *Approximation Algorithms for Multidimensional Bin Packing*. PhD thesis, Georgia Institute of Technology, USA, 2015. 9, 12

[KK82] Narendra Karmarkar and Richard M. Karp. An efficient approximation scheme for the one-dimensional bin-packing problem. In *FOCS*, pages 312–320, 1982. 4, 6, 11, 12, 14

[KK03] Hans Kellerer and Vladimir Kotov. An approximation algorithm with absolute worst-case performance ratio 2 for two-dimensional vector packing. *Oper. Res. Lett.*, 31(1):35–41, 2003. 21

[KMRW01] Yoshiharu Kohayakawa, Flávio Keidi Miyazawa, Prabhakar Raghavan, and Yoshiko Wakabayashi. Multidimensional cube packing. *Electronic Notes in Discrete Mathematics*, 7:110–113, 2001. 15

[KO07] David R. Karger and Krzysztof Onak. Polynomial approximation schemes for smoothed and random instances of multidimensional packing problems. In *SODA*, pages 1207–1216, 2007. 11

[KR00] Claire Kenyon and Eric Rémila. A near-optimal solution to a two-dimensional cutting stock problem. *Math. Oper. Res.*, 25(4):645–656, 2000. 14, 17, 19, 20

[Krö95] Berthold Kröger. Guillotineable bin packing: A genetic approach. *European Journal of Operational Research*, 84(3):645–661, 1995. 16

[KS10] Ariel Kulik and Hadas Shachnai. There is no eptas for two-dimensional knapsack. *Information Processing Letters*, 110(16):707–710, 2010. 11

[KS15] Arindam Khan and Mohit Singh. On weighted bipartite edge coloring. *To appear in FSTTCS*, 2015. 9

[KvH13] Brian Kell and Willem-Jan van Hoeve. An mdd approach to multidimensional bin packing. In *Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems*, pages 128–143. Springer, 2013. 16

[LC90] Keqin Li and Kam-Hoi Cheng. On three-dimensional packing. *SIAM Journal on Computing*, 19(5):847–867, 1990. 17

[LC92] Keoin Li and Kam Hoi Cheng. Heuristic algorithms for on-line packing in three dimensions. *Journal of Algorithms*, 13(4):589–605, 1992. 17

[LL85] C. C. Lee and D. T. Lee. A simple on-line bin-packing algorithm. *J. ACM*, 32(3):562–572, 1985. 7, 12, 14

[LM12] Shachar Lovett and Raghu Meka. Constructive discrepancy minimization by walking on the edges. In *FOCS*, pages 61–67, 2012. 7

[LMM02] Andrea Lodi, Silvano Martello, and Michele Monaci. Two-dimensional packing problems: A survey. *European Journal of Operational Research*, 141(2):241–252, 2002. 4, 14, 15

[LMV02] Andrea Lodi, Silvano Martello, and Daniele Vigo. Heuristic algorithms for the three-dimensional bin packing problem. *European Journal of Operational Research*, 141(2):410–420, 2002. 16

[Lod99] Andrea Lodi. Algorithms for two-dimensional bin packing and assignment problems. *Doktorarbeit, DEIS, Universita di Bologna*, 1999. 16

[LTW+90] Joseph YT Leung, Tommy W Tam, CS Wong, Gilbert H Young, and Francis YL Chin. Packing squares into a square. *Journal of Parallel and Distributed Computing*, 10(3):271–275, 1990. 15

[MFNK96] Hiroshi Murata, Kunihiro Fujiyoshi, Shigetoshi Nakatake, and Yoji Kajitani. Vlsi module placement based on rectangle-packing by the sequence-pair. *Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on*, 15(12):1518–1524, 1996. 16

[MHDC05] Yuchun Ma, Xianlong Hong, Sheqin Dong, and CK Cheng. 3d cbl: An efficient algorithm for general 3d packing problems. In *Circuits and Systems, 2005. 48th Midwest Symposium on*, pages 1079–1082. IEEE, 2005. 17

[MM68] Aram Meir and Leo Moser. On packing of squares and cubes. *Journal of combinatorial theory*, 5(2):126–134, 1968. 10

[MRT13] Adam Meyerson, Alan Roytman, and Brian Tagiku. Online multidimensional load balancing. In *APPROX*, pages 287–302, 2013. 22

[MV98] Silvano Martello and Daniele Vigo. Exact solution of the two-dimensional finite bin packing problem. *Management science*, 44(3):388–399, 1998. 15

[MW04] Flavio Keidi Miyazawa and Yoshiko Wakabayashi. Packing problems with orthogonal rotations. In *LATIN*, pages 359–368. Springer, 2004. 14

[NFMK98] Shigetoshi Nakatake, Kunihiro Fujiyoshi, Hiroshi Murata, and Yoji Kajitani. Module packing based on the bsg-structure and ic layout applications. *Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on*, 17(6):519–530, 1998. 16

[NNN12] Alantha Newman, Ofer Neiman, and Aleksandar Nikolov. Beck's three permutations conjecture: A counterexample and some consequences. In *FOCS*, pages 253–262, 2012. 7, 23

[NW16] Giorgi Nadiradze and Andreas Wiese. On approximating strip packing with a better ratio than 3/2. *To appear in SODA*, 2016. 17

[OPR11] Ekow Otoo, Ali Pinar, and Doron Rotem. A linear approximation algorithm for 2-dimensional vector packing. *arXiv preprint arXiv:1103.0260*, 2011. 23

[Pas03] Arfath Pasha. *Geometric bin packing algorithm for arbitrary shapes*. PhD thesis, University of Florida, 2003. 16

[PS07] David Pisinger and Mikkel Sigurd. Using decomposition techniques and constraint programming for solving the two-dimensional bin-packing problem. *INFORMS Journal on Computing*, 19(1):36–51, 2007. 15

[PST95] S.A. Plotkin, D. Shmoys, and E. Tardoso. Fast approximate algorithm for fractional packing and covering problems. *Mathematics of Operations Research*, 20:257–301, 1995. 7, 11

[PTUW11] Rina Panigrahy, Kunal Talwar, Lincoln Uyeda, and Udi Wieder. Heuristics for vector bin packing. *research. microsoft. com*, 2011. 4, 23

[Ram92] Balasubramanian Ram. The pallet loading problem: A survey. *International Journal of Production Economics*, 28(2):217–225, 1992. 3

[RBLL89] Prakash Ramanan, Donna J Brown, Chung-Chieh Lee, and Der-Tsai Lee. On-line bin packing in linear time. *Journal of Algorithms*, 10(3):305–326, 1989. 7

[Rot12] Thomas Rothvoß. The entropy rounding method in approximation algorithms. In *SODA*, pages 356–372, 2012. 7, 11

[Rot13] Thomas Rothvoß. Approximating bin packing within o(log opt * log log opt) bins. In *FOCS*, pages 20–29, 2013. 6, 7

[SBS+10] Martin Schuster, Richard Bormann, Daniela Steidl, Saul Reynolds-Haertle, and Mike Stilman. Stable stacking for the distributor's pallet packing problem. In *IROS*, pages 3646–3651. IEEE, 2010. 16

[Sei02] Steven S. Seiden. On the online bin packing problem. *J. ACM*, 49(5):640–671, 2002. 7, 15

[Sho91] Peter W Shor. How to pack better than best fit: tight bounds for average-case online bin packing. In *FOCS*, pages 752–759, 1991. 8

[Ski99] Steven Skiena. Who is interested in algorithms and why?: lessons from the stony brook algorithms repository. *ACM SIGACT News*, 30(3):65–74, 1999. 3

[SKM03] Keishi Sakanushi, Yoji Kajitani, and Dinesh P Mehta. The quarter-state-sequence floorplan representation. *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, 50(3):376–386, 2003. 16

[SP92] Paul E Sweeney and Elizabeth Ridenour Paternoster. Cutting and packing problems: a categorized, application-orientated research bibliography. *Journal of the Operational Research Society*, pages 691–706, 1992. 16

[Spi94] Frits CR Spieksma. A branch-and-bound algorithm for the two-dimensional vector packing problem. *Computers & operations research*, 21(1):19–25, 1994. 4, 22

[SS09] A Sebo and G Shmonin. Proof of the modified integer round-up conjecture for bin packing in dimension 7. *Manuscript*, 2009. 7, 23

[SSVC10] Mark Stillwell, David Schanzenbach, Frédéric Vivien, and Henri Casanova. Resource allocation algorithms for virtualized service hosting platforms. *Journal of Parallel and Distributed Computing*, 70(9):962–974, 2010. 23

[ST97] Guntram Scheithauer and Johannes Terno. Theoretical investigations on the modified integer round-up property for the one-dimensional cutting stock problem. *Operations Research Letters*, 20(2):93–100, 1997. 7, 23

[Ste97] A Steinberg. A strip-packing algorithm with absolute performance bound 2. *SIAM Journal on Computing*, 26(2):401–409, 1997. 19

[SvS02] Steven S Seiden and Rob van Stee. New bounds for multi-dimensional packing. In *SODA*, pages 486–495, 2002. 15

[SW84] SC Sarin and WE Wilhelm. Prototype models for two-dimensional layout design of robot systems. *IIE transactions*, 16(3):206–215, 1984. 4, 23

[SW05] Steven S. Seiden and Gerhard J. Woeginger. The two-dimensional cutting stock problem revisited. *Math. Program.*, 102(3):519–530, 2005. 19

[Vaz01] Vijay V Vazirani. *Approximation algorithms*. Springer Science & Business Media, 2001. 4

[VM87] D Vercruyssen and H Muller. Simulation in production. *A report of the University of Gent, Belgium*, 1987. 23

[vS15] Rob van Stee. Sigact news online algorithms column 26: Bin packing in multiple dimensions. *ACM SIGACT News*, 46(2):105–112, 2015. 15, 20

[vV92] André van Vliet. An improved lower bound for on-line bin packing algorithms. *Inf. Process. Lett.*, 43(5):277–284, 1992. 7, 23

[Woe97] Gerhard J. Woeginger. There is no asymptotic ptas for two-dimensional vector packing. *Inf. Process. Lett.*, 64(6):293–297, 1997. 20, 22

[WS11] David P Williamson and David B Shmoys. *The design of approximation algorithms*. Cambridge University Press, 2011. 4, 6, 23

[Yao80] Andrew Chi-Chih Yao. New algorithms for bin packing. *Journal of the ACM (JACM)*, 27(2):207–227, 1980. 20

[Zha05] Guochuan Zhang. A 3-approximation algorithm for two-dimensional bin packing. *Oper. Res. Lett.*, 33(2):121–126, 2005. 14