Course: MATH 4032 Combinatorial Analysis (Spring'07) – Homework 1

Instructor : Prasad Tetali, office: Skiles 234, email: tetali@math.gatech.edu
Office Hours: **Mon, Wed 11am – noon, Thurs. 10-11:00am**

Due: Friday, Jan. 19th

1. Prove the simple case of Erdős-Ko-Rado theorem: if $k = n/2$, then the number of k-sets in any intersecting family is at most $\frac{1}{2}\binom{n}{k}$.

2. Let $X = \{1, 2, \ldots, 7\}$, and let $\mathcal{B}$ be the following family of seven subsets of X:

$$\mathcal{B} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}.$$

Then let $\mathcal{F} = \{A \subset X : A$ contains B for some $B \in \mathcal{B}\}$. Show that $\mathcal{F}$ is intersecting and $|\mathcal{F}| = 2^{7-1} = 64$.

3. Let $G = (V, E)$ be a graph on n vertices and let $t(G)$ be the number of triangles in it. Show that

$$t(G) \geq \frac{|E|}{3n}(4 \cdot |E| - n^2).$$

Sketch: For an edge $e = \{x, y\}$, let $t(e)$ be the number of triangles containing e. Let $B = V \setminus \{x, y\}$. Among the vertices in B there are precisely $t(e)$ vertices which are adjacent to both x and y. Every other vertex in B is adjacent to at most one of these two vertices. We thus obtain $d(x) + d(y) - t(e) \leq n$. Summing over all edges $e = \{x, y\}$ we obtain

$$\sum_{e \in E}(d(x) + d(y)) - \sum_{e \in E} t(e) \leq n \cdot |E|.$$

Apply the Cauchy-Schwarz inequality to estimate the first sum.

Comment: This implies that a graph on an even number n of vertices with $n^2/4+1$ edges not only contains one triangle (as it must be by Mantel's theorem), but more than $n/3$.

Optional Problems.

4. Let $G = (V, E)$ be a graph. Let $d(x)$ denote the degree of x, for each $x \in V$. Explain why the following holds:

$$\sum_{x \in V} d(x)^2 = \sum_{\{x,y\} \in E} (d(x) + d(y)).$$

5. A family of subsets of X is said to be *2-colorable*, if it is possible to assign one of two colors (RED and BLUE, say) to the elements of X so that no set in the family is *monochromatic* – meaning no set should be all RED or all BLUE. Is the family $\mathcal{B}$ in Problem 2 above 2-colorable?